X. *An Experimental Determination of the Values of the Velocities of Normal Propagation of Plane Waves in different directions in a Biaxal Crystal, and a Comparison of the Results with Theory.*

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PART I.

Preliminary.—*Professor* STOKES'S *Report to the British Association,* 1862, *with Outline of the Method.*

Section I.—*Review of Previous Experiments and Criticism of* FRESNEL'S.

IN his report to the British Association in 1862, Professor STOKES says: "The exactness of the spheroidal form, assigned by HUYGHENS to the sheet of the wave surface within Iceland spar, does not seem to have been tested to the same degree of rigour as the ordinary refraction of the ordinary ray; for the methods applied by WOLLASTON (Phil. Trans., 1802, p. 381) and MALUS (Mem. de l'Institut Sav. Étran., tom. ii., p. 303, 1811) for observing the extraordinary refraction can hardly bear comparison for exactness with the method of prismatic refraction adopted for the ordinary ray; and observations on the absolute velocities of propagation in different directions within biaxal crystals are almost wholly wanting.

"This has long been recognized as a desideratum, and it has been suggested to employ for the purpose the displacement of fringes of interference.

"It seems to me that a slight modification of the ordinary method of prismatic refraction would be more convenient and exact.

"Let the crystal to be examined be cut, unless natural faces or planes of cleavage answer the purpose, so as to have two planes inclined at an angle suitable for the measure of refractions; there being at least two natural faces or cleavage planes left undestroyed, so as to permit of an exact measure of the directions of any artificial faces. The prism thus formed having been mounted as usual and placed in any azimuth, let the angle of incidence or emergence (according as the prism remains fixed or turns with the telescope) be measured by observing the light reflected from the surface, and likewise the deviation for several standard-fixed lines in the spectrum; each observation furnishes us with an angle of incidence and the corresponding angle of emergence, the angle of the prism being known. For if ϕ be the angle of incidence, D the deviation, i angle of prism, ψ angle of emergence,

$$D=\phi+\psi-i$$
$$\therefore \psi=D+i-\phi$$

"But without making any supposition as to the law of double refraction, or *assuming anything beyond the truth of* HUYGHENS's *principle*, which, following directly from the superposition of small motions, lies at the base of the whole theory of undulations, we may at once deduce from the directions of incidence and emergence the direction and velocity of propagation of the wave within the crystal. For if a plane wave be incident on any plane refracting surface, it follows directly from HUYGHENS's principle that the refracted wave or waves will be plane, and if ϕ be the angle of incidence, ϕ' the inclination of the refracted wave to the surface, V the velocity of propagation in air, v the wave velocity in the crystal

$$\frac{\sin\phi}{V}=\frac{\sin\phi'}{v}.$$

"And if ϕ' ψ' be the inclination of either refracted wave to the faces of our crystal prism we have

$$v\sin\phi=V\sin\phi' \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$v\sin\psi=V\sin\psi' \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\phi'+\psi'=i \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

adding and subtracting (1) and (2) and remembering (3), we get respectively

$$v\sin\frac{\phi+\psi}{2}\cos\frac{\phi-\psi}{2}=V\sin\frac{i}{2}\cos\frac{\phi'-\psi'}{2} \quad . \quad . \quad . \quad . \quad (4)$$

$$v\cos\frac{\phi+\psi}{2}\sin\frac{\phi-\psi}{2}=V\cos\frac{i}{2}\sin\frac{\phi'-\psi'}{2} \quad . \quad . \quad . \quad . \quad (5)$$

"By division

$$\tan\frac{\phi+\psi}{2}\cot\frac{\phi-\psi}{2}=\tan\frac{i}{2}\cot\frac{\phi'-\psi'}{2}$$

or

$$\tan\frac{\phi'-\psi'}{2}=\tan\frac{i}{2}\tan\frac{\phi-\psi}{2}\cot\frac{\phi+\psi}{2} \quad . \quad . \quad . \quad . \quad (6)$$

"Equations (3) and (6) determine ϕ' and ψ', and v is known from

$$\frac{\sin\phi}{V}=\frac{\sin\phi'}{v}$$

or

$$\frac{\sin\psi}{V}=\frac{\sin\psi'}{v}$$

" Hence we can find the velocity of propagation of the wave, the normal to which lies in a plane perpendicular to the faces of the prism, and makes known angles with those faces, and hence with the crystallographic axes."

In accordance with these suggestions, I undertook a series of observations at the Cavendish Laboratory, Cambridge, which I propose in the present paper to describe; I also wish to discuss the results arrived at, and to compare them with those deduced from FRESNEL's and some of the rival theories of double refraction.

But previous to this it seems natural to devote some space to the consideration of the experiments that have already been made with the same object.

These we may divide into two classes: (1.) Those which have reference to Iceland spar and other uniaxal crystals; (2.) Those in which biaxal crystals were used.

BREWSTER (Thirteenth Report of the British Association) proved conclusively that one wave in Iceland spar obeys strictly the ordinary law of refraction.

SWAN (Edinburgh Trans., vol. xvi., p. 375) obtained by direct measurements with prisms placed in the position of minimum deviation values of the ordinary refractive index, which differ at most by ·00002.

To verify his construction for the extraordinary ray, HUYGHENS himself made but few experiments, and it was not till 1802 that WOLLASTON (Phil. Trans., 1802, p. 381) attempted to test the theory with any degree of exactness.

In 1810 MALUS ("Théorie de la Double Refraction," Paris Mém. des Savants Étrangers, tom. ii., p. 303) undertook a series of experiments with the same object in view.

Rather later, BIOT undertook the same task, while more recently RUDBERG (POGG. Ann., vol. xiv., p. 45) and MASCART have measured the values of the principal indices by means of prisms cut parallel to the axis.

Since 1862 Professor STOKES has applied the method indicated above to prisms of Iceland spar, and finds the results of experiment agree with HUYGHENS's construction with a possible difference of ·0001 in the values of the refractive index. The details of the experiments are as yet unpublished.

(2.) We must consider next the case of biaxal crystals. Various experimenters have determined the values of the principal refractive indices for different crystals. FRESNEL alone has endeavoured to verify his theory by experiment ('Œuvres Complètes de FRESNEL,' tom. ii., p. 415; second supplement to the "Mémoire sur la Double Refraction"). The method adopted was to observe the displacement of the fringes of interference, formed from two parallel slits, produced by introducing two plates of topaz of the same thickness cut in different directions from the same crystal into the paths of the pencils proceeding from the two slits respectively, and to compare this with the displacement calculated on FRESNEL's theory.

The first set of observations combined with the angle between the optic axes serves to determine the principal velocities a, b, c. a being assumed to be unity, let

$$b^2 = a^2 - \beta = 1 - \beta$$
$$c^2 = a^2 - \gamma = 1 - \gamma$$

Then

$$\beta = 0\cdot00338$$
$$\gamma = 0\cdot01222$$
$$\gamma - \beta = 0\cdot00884$$

In the second set of observations the plates are inclined to the incident light, so that the light when in the crystal passes along an optic axis.

To calculate the displacement of the fringes on FRESNEL'S theory, it becomes necessary to know the velocity of the light in the crystal; this is given by

$$v_1^2 = 1 - \gamma \sin^2 \chi$$

χ being the angle between the direction of vibration and the normal to that circular section of the wave surface whose radius is a or unity.

Hence

$$v_1 = 1 - \tfrac{1}{2}\gamma \sin^2 \chi$$

neglecting $\gamma^2 \sin^4 \chi$, &c.

Theory gives for the displacement measured in wave lengths

$$N = 12\cdot73$$

Experiment gives

$$N = 11\cdot87$$

For the other two waves theory gives

$$N = 46\cdot03$$

Experiment gives

$$N = 45\cdot49$$

In the third experiment the light was incident at about 60°.

In this case, ϕ being the angle between the direction of vibration and the normal to the circular section radius b

$$v_1^2 = 1 - \beta \cos^2 \phi - \gamma \sin^2 \phi$$

and to the same approximation

$$v_1 = 1 - \tfrac{1}{2}(\beta \cos^2 \phi + \gamma \sin^2 \phi)$$

Whence theory gives

$$N = 19\cdot57$$

Experiment gives

$$N = 20\cdot63$$

Thus in each case the agreement between theory and experiment is fairly close; but we must remember that the quantities to be measured are very small, and that in obtaining the theoretical results, small quantities have been neglected, without showing the effects they would produce, and which, though small, may be sufficient to constitute the difference between FRESNEL'S and some rival theory. Moreover, the values of the principal refractive indices in topaz are so nearly equal that any conceivable wave-surface will differ but little from two spheres.

To estimate then the weight which we may attach to these results, let us see how far the experiments are consistent with one of the rival theories.

Now the equation to the surface of wave slowness in the theories of GREEN and CAUCHY involves more constants than the three principal refractive indices of the crystal.

To determine these we must make the principal sections agree with experimental results in more points than the extremities of the axes; and it is probable, therefore, that they will differ but little throughout.

In fact, the existence of these unnecessary constants is a radical defect in both these theories.

Lord RAYLEIGH, however, has proposed a theory in which the only constants are the principal velocities (Phil. Mag., vol. 41, series 4, 1871).

The shape of the surface of wave slowness is also considerably different from that given by FRESNEL'S theory.

Its principal sections consist of a circle and the inverse of an ellipse instead of a circle and an ellipse.

He supposes "the density of the ether in a crystal to be a function of the direction of displacement," while the forces called into play are the same as in a homogeneous medium.

From this he deduces as the equation to determine the velocity of a wave front, of which the direction cosines of the normal are l, m, n, the equation

$$\frac{a^2l^2}{V^2-a^2}+\frac{b^2m^2}{V^2-b^2}+\frac{c^2n^2}{V^2-c^2}=0\cdot$$

a, b, c being the principal velocities.

Let us find the velocity for a wave perpendicular to the plane zx, making an angle α with Ox, $l=-\sin\alpha$, $m=0$, $n=\cos\alpha$.

$$V^2(c^2\cos^2\alpha+a^2\sin^2\alpha)=a^2c^2$$

or since $a=1$, $c^2=1-\gamma$

$$V^2\{(1-\gamma)\cos^2\alpha+\sin^2\alpha\}=1-\gamma$$

$$V^2=(1-\gamma)(1-\gamma\cos^2\alpha)^{-1}$$

$$V=(1-\tfrac{1}{2}\gamma)(1+\tfrac{1}{2}\gamma\cos^2\alpha)$$

neglecting γ^2, &c.

$$V=1-\tfrac{1}{2}\gamma(1-\cos^2\alpha)$$

$$=1-\tfrac{1}{2}\gamma\sin^2\alpha$$

But replacing α by χ, this is exactly the form found above for v_1 in FRESNEL'S second experiment.

Now let us take a wave perpendicular to the plane yz inclined at an angle ϕ to Oy.

Then $l=0$, $m=\sin\phi$, $n=\cos\phi$, and the equation for V^2 becomes

$$\frac{b^2\sin^2\phi}{V^2-b^2}+\frac{c^2\cos^2\phi}{V^2-c^2}=0$$

$$V^2(b^2\sin^2\phi+c^2\cos^2\phi)=b^2c^2$$

or

$$V^2(1-\beta\sin^2\phi-\gamma\cos^2\phi)=(1-\beta)(1-\gamma)=1-(\beta+\gamma)$$

to the same approximation

$$V=\{1-\tfrac{1}{2}(\beta+\gamma)\}\{1+\tfrac{1}{2}(\beta\sin^2\phi+\gamma\cos^2\phi)\}$$

$$=1-\tfrac{1}{2}(\beta\cos^2\phi+\gamma\sin^2\phi)$$

and this agrees with the formula used in FRESNEL'S third experiment.

Thus FRESNEL'S experiments afford as much a verification of Lord RAYLEIGH'S theory as of his own, and are, therefore, an insufficient test of the truth of either.

They are, however, the only attempts made to verify the theory.

Section II.—*Description of the Apparatus used and Method of making the Experiments.*

My own work was undertaken in the hope of obtaining results sufficiently accurate to decide between some of the various theories which have been propounded.

I proceed to describe the apparatus used.

The crystal itself was a piece of arragonite, very clear and of a light colour, obtained from Germany by the Demonstrator at the Cavendish Laboratory, Cambridge, through HILGER, of Tottenham Court Road, London, W.C.

Originally it was in the form of an hexagonal prism.

Four of the faces were those marked m in Professor MILLER'S mineralogy, the other two were the faces marked a.

The natural ends had been cut off, and the artificial ends were approximately perpendicular to the m and a faces.

The crystal consisted of several twins, the twin planes being parallel to two of the m faces.

At this stage Professor STOKES kindly undertook to examine the crystal, to determine in which direction to cut it with the greatest advantage for the purposes of the experiment. He found that fair reflexions were obtained from three of the m faces, and decided that they should be left untouched to determine the position of any other cut faces.

Again, the mean axis of elasticity bisects the angle between the contiguous m faces;

the optic axes lie in the plane perpendicular to this axis. A prism formed with its refracting edge nearly parallel to the axis of b, would permit of observations being made in a plane passing nearly through the optic axes; and it was found that, by inclining the faces so that the axis of a nearly bisected the angle of the prism, and making that angle about 42°, light could be passed through the crystal in the direction of either optic axis. It was therefore thought right that the prism should be so cut.

Moreover, by polishing one of the a faces and cutting a face nearly coincident with the fourth m face, a second prism was formed with its refracting edge nearly parallel to c.

It was important to get observations in different zones from the same piece of arragonite, for its chemical constitution is more or less variable, and the constants—*i.e.*, the principal refractive indices—may vary in different specimens.

The work of cutting was successfully performed by HILGER, and the faces of the prisms were very fairly plane.

The crystal being thus cut, it remained to determine accurately the position of the cut faces with reference to the crystallographic axes.

For this purpose, and throughout the observations, I used a goniometer, by GRUBB, kindly lent me by Professor STOKES.

The collimator is fixed to a graduated circle of about 9 inches diameter. The graduations, which are in silver, are on the *flat face*, not the limb of the circle, and the circle is divided to arcs of 10′.

Attached to the reading telescope there are two verniers; the *graduations on the verniers being in the same plane* as those of the circle; and by means of these the circle can be read to angles of 10″.

In the centre of this circle there is a table to hold the prism; the table being attached to a second graduated circle with verniers which can be read to half minutes.

It was requisite throughout the experiments to have the edge of the prism parallel to the axis of revolution of the telescope.

To attain this the prism was attached to a stand, adjustable with set screws, which rested in slots on the table of the goniometer.

The reading telescope was fitted with a needle point instead of the usual cross wires.

In order to level the prism, I covered half the slit, so that when viewed directly the line of junction of the covered and uncovered portions coincided with the needle point.

I then moved the telescope until the images of the slit, formed by reflexion at the faces of the prism, were brought in turn into the field; and altered the levelling screws till the same coincidence was obtained.

Homogeneous light was secured by using a BUNSEN's burner with a platinum wick, one end of which was immersed in a saturated solution of salt-and-water.

By the advice of Professor STOKES, the observations were made at angles of incidence increasing in arithmetic progression.

The direct reading was taken several times during each day's work to avoid error.

The method of observing was as follows:—I set the telescope to a given scale reading, and turned the prism till the image of the slit seen by reflexion at one of its faces coincided with the needle point.

The difference between the direct reading and the known one to which the telescope was set gave the supplement of twice the angle of incidence, whence the angle of incidence is easily obtained. I then turned the telescope till one image of the slit seen by refraction through the prism coincided with the needle point and took the reading.

The difference between this and the direct reading gave the deviation for that wave. The observation was then repeated for the second wave.

I then reset the telescope, so that its axis made an angle of 4° with its former position, and took a similar set of observations. This gave a series of deviations corresponding to angles of incidence increasing by 2°.

Just in the neighbourhood of the optic axes, observations were made at every 1° of incidence.

For the first prism, each observation in the set finally chosen was repeated three times, and the mean of the results taken.

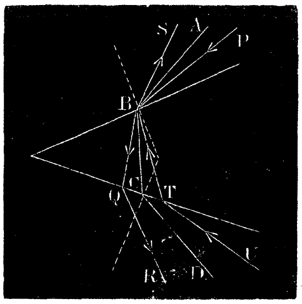
The difference between this mean and any observation was only in a few cases as great as 10″.

In the case of the second prism, the observed values of the deviation on different days for the same angle of incidence agreed so closely, rarely differing by more than 20″, that it seemed unnecessary to take a third set.

I found also that the refracted images were much more distinct for angles of incidence greater than that for minimum deviation, than for angles less.

On passing through the position of minimum deviation, I therefore reversed the prism, so that the face of incidence became that of emergence.

Fig. 1.



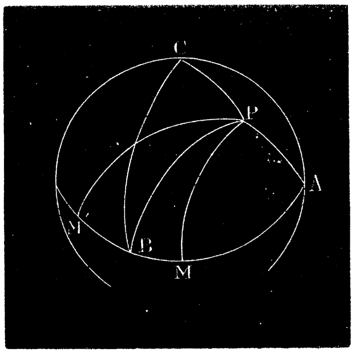
Thus, if A B C D be the ray undergoing minimum deviation, for a ray such as P B Q R, I made P B the incident ray, Q R the emergent; while for a ray like S B T U, I reversed the prism, so that U T became the incident, B S the emergent ray.

The cause of the increased distinctness lies in the fact that as the angle of incidence increases from that giving minimum deviation, the breadth of the image of the slit decreases, and *vice versâ*.

Section III.—*Determination of the Position of the Principal Plane of the First Prism with reference to the Crystallographic Axes.—Values of the Quantities observed in the Experiments, and Calculation of the Reciprocal of the Wave Velocity for the First Prism.*

We proceed now to the measurements made to determine the position of the principal plane of the first prism, of which the faces are P $P_,$.

Fig. 2.



Let A B C be the points in which the axes of the crystal cut a sphere, with its centre at the origin.

Let M M′ be the poles of the *m* faces of the crystal, M M′ lie in the great circle A B, and the angle M M′ is known; also B M=B M′; let P $P_,$ be the poles of the cut faces of the prism.

Let

$$PM=\theta,\ PM'=\theta'$$

$$P_,M=\theta_,,\ P_,M'=\theta'_,$$

Let α, β, γ be the angles which O P makes with the axes.

Let $MA=\mu$.

From triangle P A M,

$$\cos PM=\cos AP\cos AM+\sin AP\sin AM\cos PAM$$

or

$$\cos\theta=\cos\alpha\cos\mu+\sin\alpha\sin\mu\cos A$$

From P A M′,

$$\cos\theta'=-\cos\alpha\cos\mu+\sin\alpha\sin\mu\cos A$$

$$\therefore\cos\theta-\cos\alpha\cos\mu=\cos\theta'+\cos\alpha\cos\mu$$

$$\therefore\cos\alpha=\frac{\cos\theta-\cos\theta'}{2\cos\mu}$$

$$=\frac{\sin\frac{\theta+\theta'}{2}\sin\frac{\theta'-\theta}{2}}{\cos\mu}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Again, from triangle P M B,

$$\cos PM = \cos BP \cos BM + \sin BP \sin BM \cos PBM$$

or

$$\cos \theta = \cos \beta \sin \mu + \sin \beta \cos \mu \cos B$$

and from P B′ M,

$$\cos \theta' = \cos \beta \sin \mu - \sin \beta \cos \mu \cos B$$

$$\therefore \cos \beta = \frac{\cos \theta + \cos \theta'}{2 \sin \mu}$$

$$= \frac{\cos \frac{\theta+\theta'}{2} \cos \frac{\theta'-\theta}{2}}{\sin \mu} \quad \ldots \ldots \ldots \ldots \quad (2)$$

Now the angles θ θ' were determined by observation.

Each angle was observed six times on two different days.

The mean of the results was,

$$\theta = 77° \ 18' \ 44''$$
$$\theta' = 100° \ 43' \ 30''$$

Some difficulty was experienced about the angle μ.

Professor MILLER gives it as 58° 5′; but this, when substituted in the above formulæ, gives a value for $P P_{,}$ the supplement of the angle of the crystal, which differs considerably from the measured angle.

I therefore measured the angle μ and found it to be 58° 30′ nearly.

Mr. GARNETT (the Demonstrator at the Cavendish Laboratory) kindly communicated this result to the Rev. H. P. GURNEY, of Clare College, who was then lecturing for Professor MILLER, and he measured the angle with very nearly the same result as I had already arrived at. Taking, then, the value he obtained, and assuming as before that O B bisects the angle M O M′, we have

$$\mu = 58° \ 28'.*$$

Whence from the formulæ (1) (2)

$$\alpha = 67° \ 10' \ 35'' \quad \ldots \ldots \ldots \ldots \quad (3)$$

$$\beta = 88° \ 52' \ 40'' \quad \ldots \ldots \ldots \ldots \quad (4)$$

For the position of $P_{,}$ we have similarly,

$$\theta_{,} = 80° \ 39'$$
$$\theta_{,}' = 101° \ 16' \ 23''$$

Whence,

$$\alpha_{,} = 69° \ 59' \ 20'' \quad \ldots \ldots \ldots \ldots \quad (5)$$

$$\beta_{,} = 91° \ 6' \ 30'' \quad \ldots \ldots \ldots \ldots \quad (6)$$

* This value of the angle μ is given in DANA's 'Mineralogy,' for specimens of arragonite coming from Silesia. The piece used was obtained from that neighbourhood.

To determine the angles B A P, B A $P_{,}$, the triangles B A P and B′ A $P_{,}$ give $\left(\text{since } AB=\frac{\pi}{2}\right)$

$$\cos PAB=\frac{\cos\beta}{\sin\alpha}$$

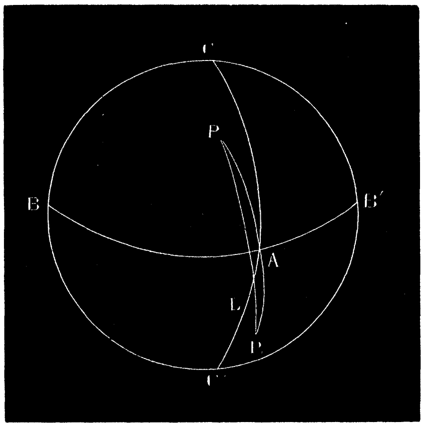
whence

$$BAP=88°\ 46'\ 40''$$

similarly from B′ A $P_{,}$

$$B'AP_{,}=88°\ 49'\ 10''$$

Fig. 3.



From triangle P A $P_{,}$

$$\cos PP_{,}=\cos AP_{,}\cos AP+\sin AP_{,}\sin AP\cos PAP_{,}$$

Whence

$$PP_{,}=137°\ 9'\ 55''$$

The mean of numerous experiments gave

$$PP_{,}=137°\ 9'\ 30''$$

The smallness of the difference affords a strong presumption in favour of the measurements.

Let P $P_{,}$ cut A C′ in L.

Then

$$\sin AP_{,}P=\frac{\sin PAP_{,}}{\sin PP_{,}}\sin AP$$

Whence

$$AP_{,}P=0°\ 3'\ 23'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Again

$$\cot LP_{,}\sin AP_{,}=\cos AP_{,}\cos AP_{,}L+\sin AP_{,}L\cot LAP_{,}$$

$$\therefore LP_{,}=67°\ 27'\ 25''. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\sin ALP=\frac{\sin AP_{,}\sin LAP_{,}}{\sin LP_{,}}$$

$$ALP=1°\ 12'\ 4'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\sin \text{AL} = \frac{\sin \text{AP}_{,} \sin \text{AP}_{,}\text{L}}{\sin \text{ALP}_{,}}$$

$$\text{AL} = 2° \, 31' \, 58'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The solution of the triangle A L P leads to the same results.

We now come to the experimental determination of the deviation and the calculations of the refractive indices.

Throughout the work for this prism, ϕ represents the angle between the normal to the wave incident on or emerging from the face $\text{P}_{,}$ when in air, ϕ' the corresponding angle in the crystal, while ψ ψ' are the angles for the face P. I propose then to give a table of the values of ϕ or ψ according as the wave was incident on $\text{P}_{,}$ or P.

(2.) Of $\text{D}+i$, i being the angle of the prism, $\text{D}+i$ is given because the expression $\frac{\phi+\psi}{2}$, which is equal to $\frac{\text{D}+i}{2}$, occurs in the formula, and it saved trouble to calculate $\text{D}+i$ at once from the readings.

(3.) Of the values of ϕ' deduced from the formulæ

$$\tan \frac{\phi' \sim \psi'}{2} = \tan \frac{i}{2} \tan \frac{\phi \sim \psi}{2} \cot \frac{\phi+\psi}{2}$$

proved in the first section and $\phi'+\psi'=i$.

(4.) Of the values of μ the refractive index as the mean of the results, given by the two formulæ

$$\sin \phi = \mu \sin \phi'$$
$$\sin \psi = \mu \sin \psi'$$

Any difference in the two values of μ found from these two formulæ was due only to the errors necessarily arising from the use of proportional parts, and rarely exceeded ·00001.

I also introduce a list of errors in the calculated quantities which would occur from errors of $10''$ in each of the observed quantities, taken so as to produce the maximum effect in the result.

This list was obtained by setting down from the tables the errors to which the supposed errors in D and ϕ would give rise.

The numbers in the table of errors for μ are the digits in the fifth place of decimals.

The next page gives the complete work for one observation as a sample.

Work for reducing one Observation.

Direct reading	263° 39′
Reflexion reading	180° 5′
Deviation reading	297° 34′ 12″
i=angle of prism	42° 52′ 30″

$$\phi=\tfrac{1}{2}\{180-(\text{Direct}-\text{Reflexion})\}$$
$$=\tfrac{1}{2}(180^\circ\ 5'-83^\circ\ 39')$$
$$=48^\circ\ 13'.\quad \text{Error}-5''.$$

$$\text{D}+i=\text{D}-220^\circ\ 48'\ 30''$$
$$=76^\circ\ 45'\ 42''.\quad \text{Error}+10''.$$

$$\psi=\text{D}+i-\phi=28^\circ\ 32'\ 42''.\quad \text{Error}+15''.$$

$$\frac{\phi+\psi}{2}=\frac{\text{D}+i}{2}=38^\circ\ 22'\ 51''.\quad \text{Error}+5'.$$

$$\frac{\phi-\psi}{2}=\frac{\phi+\psi}{2}-\psi=9^\circ\ 50'\ 9''.\quad \text{Error}-10''.$$

$$\text{L}\tan\frac{i}{2}=9{\cdot}5936354.$$

L cot $\frac{\phi+\psi}{2}$	Error.	L tan $\frac{\phi-\psi}{2}$	Error.	L tan $\frac{\phi'-\psi'}{2}$	Error.
10·1012497	216	9·2389842	−1252	8·9338693	−1036
$\frac{\phi'-\psi'}{2}$ 4° 54′ 30″	−4″	ϕ' 26° 19′ 45″	−4″	ψ' 16° 30′ 45″	+4″
L sin ϕ 9·8725466	−94	L sin ϕ' 9·6469205	−170	log μ ·2256261	−76
L sin ψ 9·6792905	580	L sin ψ' 9·4536616	284	log μ ·2256289	396
μ from ϕ 1·68123	−3	μ from ψ 1·68124	+15	mean 1·681235	

TABLE giving results of experiments on first prism in the first position.

Outer sheet.					Inner sheet.						
Table VI.		Table V.		Table IV.	Table I.		Table II.		Table III.		
$\delta\mu_2$.	μ_2.	$\delta\phi'$.	ϕ'.	$D+i$.	ϕ.	$D+i$.	ϕ'.	$\delta\phi'$.	μ_1.	$\delta\mu_1$.	
		″	° ′ ″	° ′ ″	° ′	° ′ ″	° ′ ″	″			
9	1·68500	6	16 17 45	77 4 20	28 13	76 50 16	16 20 2	6	1·68119	9	1
	1·68540		17 22 26	76 39 45	30 13	76 24 53	17 25 8		1·68119		2
	1·68568		18 26 14	76 21 43	32 13	76 6 30	18 29 16		1·68125		3
	1·68578		19 29 9	76 9 10	34 13	75 54 30	19 32 21		1·68134		4
	1·68589		20 30 55	76 2 30	36 13	75 47 30	20 34 27		1·68130		5
5	1·68568	5	21 31 50	75 59 46	38 13	75 45 51	21 35 22	5	1·68129	5	6
	1·68558		22 31 23	76 2 26	40 13	75 48 58	22 35 3		1·68128		7
	1·68517		23 29 57	76 8 41	42 13	75 56 28	23 33 29		1·68119		8
	1·68467		24 27 12	76 19 8	44 13	76 8 43	24 36 22		1·68123		9
	1·68421		25 22 58	76 34 3	45 13	76 24 55	25 25 56		1·68116		10
3	1·68376	4	26 17 11	76 53 13	48 13	76 45 42	26 19 45	4	1·68123	3	11
	1·68302		27 10 5	77 15 40	50 13	77 10 5	27 12 4		1·68113		12
	1·68226		28 1 18	77 42 10	52 13	77 38 48	28 2 33		1·68114		13
	1·68190		28 26 13	77 57 3	53 13	77 54 33	28 27 10		1·68104		14
	1·68153		28 50 41	78 12 53	54 13	78 11 33	28 51 13		1·68106		15
3	1·68131	4	29 14 32	78 30 12	55 13	78 29 15	29 14 54	4	1·68103	3	16
	1·68123		29 37 41	78 49 2	56 13	78 47 35	29 38 16		1·68073		17
	1·68122		30 0 16	79 9 3	57 13	79 6 25	30 1 20		1·68032		18
	1·68122		30 22 19	79 30 8	58 13	79 26 34	30 23 47		1·67996		19
	1·68114		31 4 55	80 15 15	60 13	80 9 33	31 7 20		1·67918		20
2	1·68125	3	31 45 53	81 5 10	62 13	80 56 36	31 48 39	3	1·67841	2	21
	1·68117		32 23 6	81 58 55	64 13	81 48 26	32 27 48		1·67756		22
	1·68123		32 58 34	82 57 30	66 13	82 44 26	33 4 33		1·67674		23
	1·68122		33 31 38	84 0 23	68 13	83 44 56	33 38 51		1·67591		24
	1·68134		34 1 56	85 8 18	70 13	84 50 20	34 10 28		1·67519		25
2	1·68130	3	34 29 48	86 20 25	72 13	86 0 32	34 39 24	3	1·67450	2	26
	1·68121		34 55 0	87 37 10	74 13	87 15 43	35 5 31		1·67388		27
	1·68127		35 17 10	88 59 20	76 13	88 35 51	35 28 48		1·67328		28
	1·68121		35 36 39	90 26 0	78 13	90 1 2	35 49 8		1·67274		29

ϕ is the same for both sheets.

TABLE giving results of experiments on first prism in the second position.

Outer sheet.					Inner sheet.						
Table XII.		Table XI.		Table X.	Table VII.		Table VIII.		Table IX.		
$\delta\mu_2$.	μ_2.	$\delta\phi'$.	ϕ'.	$D+i$.	ψ.	$D+i$.	ϕ'.	$\delta\phi'$.	μ_1.	$\delta\mu_1$.	
8	1·68502	″ 6	° ′ ″ 23 24 20	° ′ ″ 76 8 53	° ′ ″ 33 47 5	° ′ ″ 75 16 5	° ′ ″ 23 31 25	″ 6	1·68117	8	1
	1·68539		22 32 28	76 1 57	35 47 5	75 48 24	22 29 20		1·68122		2
	1·68570		21 31 43	75 59 53	37 47 5	75 45 32	21 28 7		1·68120		3
	1·68574		20 31 58	76 1 53	39 47 5	75 47 40	20 28 9		1·68119		4
	1·68569		19 33 28	76 8 17	41 47 5	75 54 34	19 29 32		1·68123		5
5	1·68560	5	18 36 17	76 19 0	43 47 5	76 5 45	18 32 16	5	1·68125	5	6
	1·68541		17 40 30	76 33 43	45 47 5	76 21 18	17 36 30		1·68125		7
	1·68517		16 46 12	76 52 38	47 47 5	76 41 0	16 42 17		1·68126		8
	1·68480		15 53 18	77 15 2	49 47 5	77 4 48	15 49 42		1·68126		9
	1·68439		15 2 14	77 41 52	51 47 5	77 32 47	14 58 49		1·68122		10
3	1·68386	4	14 12 45	78 12 16	53 47 5	78 4 34	14 9 48	4	1·68122	3	11
	1·68333		13 25 9	78 46 47	55 47 5	78 40 55	13 21 49		1·68131		12
	1·68277		12 39 29	79 25 24	57 47 5	79 21 4	12 37 43		1·68127		13
	1·68223		11 55 55	80 7 57	59 47 5	80 5 40	11 54 50		1·68130		14
	1·68204		11 35 2	80 31 30	60 47 5	80 29 40	11 34 15		1·68140		15
				80 55 0	61 47 5	80 55 0	11 13 55		1·68122		16

ψ is the same for both sheets.

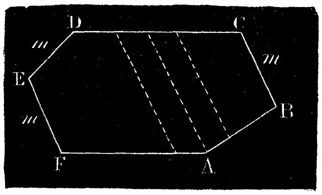
Thus Tables III., VI., IX., and XII. give the values of the refractive indices, or reciprocals of the wave velocity, for the different directions in the crystal given by II., V., VIII., and XI.

We leave here the experimental work for the first prism, and consider the second; reserving for the future the comparison of our results with theory.

Section IV.—*Description of the Second Prism: Position of its Axes.—Observed Values of the Deviations and Angles of Incidence, and Calculation of the Reciprocals of the Wave Velocity.*

It will be remembered that, in describing the cutting of the crystal, I stated that a second prism was formed with its edge nearly parallel to C, and its faces not differing much from the *m* and *a* faces of the crystal.

Fig. 4.



Let the figure, fig. 4, represent a section of the crystal by a plane perpendicular to the *m* faces.

The traces of the faces of the second prism are A B and C D.

The greater part of the face A B lies in the twin crystal, so that the results for the second prism apply to that crystal.

On examination, the *m* face B C was seen to consist of two parts, which are not in the same plane.

Observation showed that the upper half (the prism being placed as in the figure) is in the same zone as the faces D E, E F, while the lower portion is considerably removed from this zone.

I found also that when the faces D E, E F, were levelled, the line of junction of the dark and bright portions of the slit, when seen by reflexion from A B, coincided with the needle point; so that A B is in the same zone as the *m* faces; while the reflexion from C D, though in the field, was too high. Thus if Q $Q_{,}$, fig. 5, be the points where the normal to A B, fig. 4, and that to C D, fig. 4, produced backwards meet the sphere of reference, A, B, C, fig. 5, being the poles of the principal planes; Q lies on the great circle A M B; $Q_{,}$ is slightly below it.

Let M, $M_{,}'$, fig. 5, be the points where the normal to E F, and that to E D produced backwards cut the sphere. Then

$$AM=AM_{,}'=58^\circ\ 28'.$$

The mean of several observations gave for $MQ=103^\circ\ 58'$. Hence

$$AQ=45^\circ\ 30' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The mean of the observations for M $Q_{,}$ and $M_{,}'$ $Q_{,}$, gave

$$MQ_{,}=57° \ 51' \ 30''=\theta \text{ say};$$
$$M_{,}'Q_{,}=59° \ 9'=\theta' \text{ say};$$

and the triangles M A Q, M,'A Q, give,

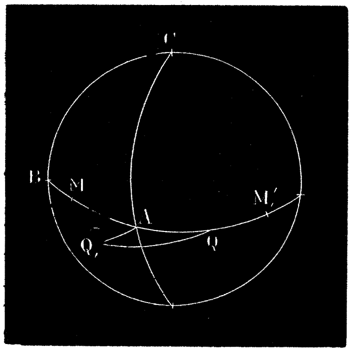
$$\cos AQ_{,}=\frac{\cos\theta+\cos\theta'}{2\cos\mu}$$

where $\mu=AM=M_{,}'A$.

Whence substituting for θ and θ'

$$AQ_{,}=2° \ 43' \ 20'' \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (2)$$

Fig. 5.



In the triangle M A Q, the three sides are known; we can thence find the angle M A Q, and obtain

$$MAQ_{,}=76° \ 24'' \ 45'. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (3)$$

In the triangle Q A Q, the sides A Q, A Q, and the angle Q A Q, are known

$$AQ = \ 45° \ 30'$$
$$AQ_{,}= \ 2° \ 43' \ 20''$$
$$QAQ_{,}=103° \ 35' \ 15''$$

Whence from the formula

$$\cos QQ_{,}=\cos AQ\cos AQ_{,}+\sin AQ\sin AQ_{,}\cos QAQ_{,}$$

we get

$$QQ_{,}=46° \ 12' \text{ (or rather less).}$$

The mean of several experiments had already given for the value of QQ,

46° 10′

These results, differing as they do by less than 2′, afford a verification of the accuracy of the measurements.

The position of the plane $Q\,Q_{,}$ is completely determined by the arc A Q and the angle $A\,Q\,Q_{,}$; we have already found,

$$AQ=45^{\circ}\ 30'$$

For the angle $A\,Q\,Q_{,}$ the triangle $Q\,A\,Q_{,}$ gives

$$\sin AQQ_{,}=\frac{\sin AQ_{,}}{\sin QQ_{,}}\sin QAQ_{,}$$

whence

$$AQQ_{,}=4^{\circ}\ 37'\ 20'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Thus we see that the principal section of the second prism is inclined to the plane c at an angle of 4° 37′ 20″.

I mentioned above that the second prism lay in the twin crystal. This was proved by passing light through the crystal, in a plane nearly parallel to the principal plane B C, when four images of the slit appeared. On cutting off the end containing the second prism, two images vanished; on cutting off the other end, only the second pair vanished, and the first pair reappeared. The work for the first prism was done with the second pair, the first pair being rarely sufficiently distinct for measurement.

Throughout the work, $\phi\ \phi'$ denote the angles which the normals to the incident or emergent wave, and the wave in the crystal make with the normal to the face Q.

The same remarks as to the reversal of the crystal hold as in the case of the first prism.

TABLE giving the results of experiments on the second prism.

Inner sheet.						Outer sheet.					
Table XV.		Table XIV.		Table XIII.		Table XVI.	Table XVII.		Table XVIII.		
$\delta\mu_1$.	μ_1.	$\delta\phi'$.	ϕ'.	$D+i$.	ϕ.	$D+i$.	ϕ'.	$\delta\phi'$.	μ_2.	$\delta\mu_2$.	
		″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″			
5	1·53026	5	22 45 17	73 44 40	36 17 30	82 54 20	20 35 20	5	1·68312	5	1
	1·53036		23 53 8	73 45 50	38 17 30	82 42 40	21 36 0		1·68330		2
	1·53031		24 59 51	73 50 40	40 17 30	82 37 0	22 35 30		1·68336		3
	1·53028		26 5 11	73 59 30	42 17 30	82 37 20	23 33 37		1·68347		4
	1·53029		27 9 1	74 12 20	44 17 30	82 43 10	24 30 19		1·68357		5
3	1·53021	4	28 11 24	74 28 40	46 17 30	82 53 50	25 25 35	4	1·68363	3	6
	1·53025		29 11 59	74 49 10	48 17 30	83 9 50	26 19 10		1·68378		7
	1·53033		30 10 46	75 13 30	50 17 30	83 30 10	27 11 11		1·68381		8
	1·53036		31 7 42	75 41 30	52 17 30	83 55 25	28 1 21		1·68391		9
	1·53037		32 2 41	76 13 15	54 17 30	84 25 15	28 49 40		1·68400		10
2	1·53040	3	32 55 37	76 48 50	56 17 30	84 59 15	29 36 10	3	1·68401	2	11
	1·53041		33 46 19	77 28 25	58 17 30	85 38 15	30 20 28		1·68413		12
	1·53043		34 34 40	78 12 5	60 17 30	86 21 40	31 2 40		1·68421		13
	1·53045		35 20 36	78 59 50	62 17 30	87 9 50	31 42 34		1·68437		14
	1·53049		36 3 56	79 52 0	64 17 30						15

ϕ is the same for both sheets.

EXPERIMENTAL results for second prism, second position.

Inner sheet.						Outer sheet.					
Table XXI.		Table XX.		Table XIX.		Table XXII.	Table XXIII.		Table XXIV.		
$\delta\mu_1$.	μ_1.	$\delta\phi'$.	ϕ'.	$D+i$.	ψ.	$D+i$.	ϕ'.	$\delta\phi'$.	μ_2.	$\delta\mu_2$.	
		″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″			
8	1·53027	6	26 4 53	73 59 25	31 42 30	83 53 55	27 58 54	6	1·68388	8	1
	1·53019		24 54 17	73 49 40	33 42 30	83 28 20	26 55 30		1·68376		2
	1·53032		23 44 57	73 45 20	35 42 30	83 1 10	25 53 10		1·68368		3
	1·53027		22 36 37	73 44 50	37 42 30	82 46 5	24 51 53		1·68344		4
	1·53017		21 29 28	73 48 15	39 42 30	82 38 0	23 51 53		1·68333		5
5	1·53018	5	20 23 47	73 56 10	41 42 30	82 36 10	22 53 14	5	1·68330	5	6
	1·53010		19 19 24	74 7 35	43 42 30	82 39 40	21 55 57		1·68320		7
	1·53023		18 16 48	74 23 30	45 42 30	82 48 40	21 0 7		1·68317		8
	1·53028		17 15 44	74 42 55	47 42 30	83 1 50	20 5 38		1·68295		9
	1·53013		16 16 9	75 5 30	49 42 30	83 20 45	19 13 4		1·68289		10
3	1·53017	4	15 18 40	75 32 25	51 42 30	83 43 40	18 21 59	4	1·68269	3	11
	1·53010		14 22 57	76 2 45	53 42 30	84 11 40	17 32 54		1·68264		12
	1·53013		13 29 24	76 37 15	55 42 30	84 44 10	16 45 43		1·68256		13
	1·53010		12 37 57	77 15 30	57 42 30	85 21 30	16 0 36		1·68256		14
	1·53003		11 48 45	77 57 40	59 42 30	86 3 10	15 17 33		1·68254		15
3	1·53010	4	11 2 10	78 44 25	61 42 30	86 48 50	14 36 29	4	1·68231	3	16
	1·53010		10 18 0	79 35 10	63 42 30	87 39 40	13 57 54		1·68237		17
	1·53017		9 36 35	80 30 30	65 42 30	88 35 45	13 21 57		1·68240		18
	1·53003		8 57 33	81 29 35	67 42 30	89 35 25	12 48 1		1·68220		19
	1·53014		8 21 52	82 34 10	69 42 30	90 40 30	12 16 53		1·68218		20

ψ is the same for both sheets.

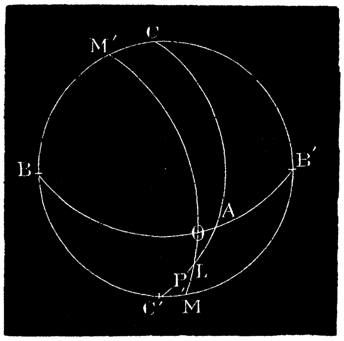
Section V.—*Determination of the Values of the Principal Indices.—Angle between the Optic Axes.*

Our next step is to determine the values of the refractive indices on FRESNEL'S theory.

It has been shown (Section III.) that the principal plane of the first prism cuts the plane C A C′ in a line inclined at an angle of 2° 31′ 58″ to O A, and makes with that plane an angle of 1° 12′ 4″.

Let us consider the intersection of this plane and the plane B C, B′ C′.

Fig. 6.



It will be a line, M O M′, say, inclined at 1° 12′, or rather less to C O C′.

Take O A, O B, O C, as axes of x, y, z, and let μ_a, μ_b, μ_c be the principal refractive indices, and therefore the semi-axes of the surface of wave slowness.

The sections of the surface of wave slowness on FRESNEL'S theory by the plane B O C are a circle of radius μ_a and an ellipse axes μ_b, μ_c in the directions of O C, O B respectively.

Hence the values of the radius vector in this direction given by experiment—*i.e.*, the values of the refractive indices for light traversing the crystal in this direction—are μ_a, and the radius vector of an ellipse axes μ_b, μ_c, inclined at about 1° 12′ to μ_b.

To determine which of the experimental values of μ is in this direction we require to find L M, L being the point of intersection of M M′ and A C′.

From the triangle C′ L M, since C′ is a right angle,

$$\cos C'LM = \tan C'L \cot LM$$

Whence

$$LM = 87° \; 28' \; 4'' \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

And if O P, be as before normal to the face P, of the first prism,

$$LP_{,} = 67° \; 27' \; 25'' \quad \text{[Section III. (8)].}$$

Whence

$$MP_{,} = 20° \; 0' \; 39'' \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

and $MP_{,}$ measures the angle between the wave normal required in the crystal and the normal to the face $P_{,}$.

This is the angle we have denoted by ϕ'.

From Tables II., III., VIII., and IX. we have corresponding values of

ϕ'	μ
19° 32′ 21″	1·68134
20° 34′ 27″	1·68130
20° 28′ 9″	1·68119
19° 29′ 32″	1·68123

Other observations gave

ϕ'	μ
19° 39′ 22″	1·68126
20° 41′ 22″	1·68129

Again, none of the experimental values of μ in Table III., lines 1 to 11, and Table IX., differ greatly from 1·68125, and this is not far from the mean of the six observations recorded above.

We take, then, this as the radius vector of an ellipse axes μ_b, μ_c, inclined at 1° 12′ to μ_b.

To determine μ_c we must have recourse to the second prism.

Did the plane of the second prism coincide with that of $x\, y$, the values of μ in Table XXI. would all be equal to μ_c.

We shall consider later the effect of the inclination of the plane of the prism to that of $x\, y$, but for the present may remark that the values in Table XXI. differ little from 1·53013, the value of μ_c found by RUDBERG.

Let us take therefore

$$\mu_c = 1{\cdot}53013 \quad \ldots \quad (3)$$

We have then in the plane B O C

$$\frac{1}{\mu^2} = \frac{\cos^2\theta}{\mu_b^2} + \frac{\sin^2\theta}{\mu_c^2}$$

θ being the angle between μ and μ_b.

Now

$$\mu = 1{\cdot}68125$$
$$\mu_c = 1{\cdot}53013$$
$$\theta = 1^\circ\, 12'$$

Whence

$$\mu_b = 1{\cdot}68132 \quad \ldots \quad (4)$$

The value found by RUDBERG is 1·68157.

The difference is considerable, being ·00025.

We may note, however, that his values of μ_b deduced from two different pieces of crystal differed by ·0004.

Now μ_a is the value of the refractive index for the first prism for the wave E when

$$\phi'=20° \ 0' \ 39''$$

The values of μ at about 30′ on either side of this are

1·68578	Table VI., 1·68589
1·68574	Table XII., 1·68569

The value

$$\mu_a=1{\cdot}68580 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

seems to represent these fairly. Let us take it at present, and see how far it gives consistent results.

RUDBERG gives

$$\mu_a=1{\cdot}68589$$

To verify these results let us find the angle between the optic axes.

If $2\delta'$ be this angle in the crystal

$$\tan\delta'=\sqrt{\left\{\frac{\dfrac{1}{\mu_b^2}-\dfrac{1}{\mu_a^2}}{\dfrac{1}{\mu_c^2}-\dfrac{1}{\mu_b^2}}\right\}}$$

Whence

$$\delta'=9° \ 4' \ 58'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and if 2δ be the angle in air as seen through a face normal to the least axis of the crystal

$$\sin\delta=\mu_b\sin\delta'$$

whence

$$\delta=15° \ 23' \ 30'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

KIRCHOFF found 15° 27′ as the result of experiments.

The value given by RUDBERG's values of the indices is

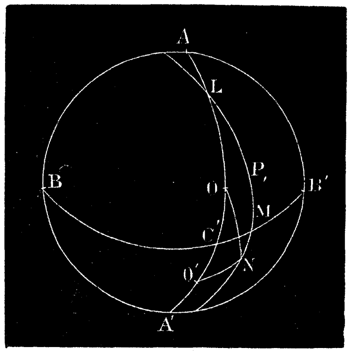
$$\delta=15° \ 7'$$

which differs from KIRCHOFF's result by 20′, whereas the value given above as the result of my experiments differs only by 3′ 30″.

Section VI.—*Calculation of the Reciprocal of the Velocity of Wave Propagation on* FRESNEL'S *Theory for the First Prism and Comparison with Experiment.—The same for Lord* RAYLEIGH'S *Theory.*

We are now in a position to calculate the reciprocals of the velocity of wave propagation for the first prism on the theory of FRESNEL.

Fig. 7.



Let O O′, fig. 7, be the points in which the optic axes cut a unit sphere.

L M the trace of the principal plane of the prism cutting A C′ A′ in L, B C′ B′ in M.

Let N be any point in the plane, in which a wave normal meets the unit sphere.

Let

$$NO=\theta \quad NO'=\theta'.$$

$$C'LM=\chi=1° \ 12' \quad \text{[Section III. (9)]}.$$

$$\cos\theta=\cos LN\cos OL+\sin LN\sin OL\cos\chi=\frac{\cos OL}{\cos\lambda}\cos(LN-\lambda)$$

if $\tan\lambda=\tan OL\cos\chi$.

Similarly

$$\cos\theta'=\frac{\cos O'L}{\cos\lambda'}\cos(LM-\lambda')$$

if $\tan\lambda'=\tan O'L\cos\chi$.

Now

$$AO=90°-9° \ 4' \ 58'' \quad \text{[Section V. (6)]}.$$
$$=80° \ 55' \ 2''$$

$$AL=2° \ 31' \ 58'' \quad \text{[Section III. (10)]}.$$

$$OL=78° \ 23' \ 4'' \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (1)$$

$$O'L=96° \ 33' \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (2)$$

whence

$$\lambda = 78^\circ\ 22'\ 55'' \quad . \quad . \quad . \quad . \quad (3)$$

$$\lambda' = 96^\circ\ 33'\ \ 5'' \quad . \quad . \quad . \quad . \quad (4)$$

Also $P_,$ being the point in which the normal to the face $P_,$ meets the sphere

$$\begin{aligned} LN &= LP_, + P_,N \\ &= LP_, + \phi' \end{aligned}$$

$$LP_, = 67^\circ\ 27'\ 25'' \quad \text{[Section III. (8)].}$$

$$\begin{aligned} LN - \lambda &= LP_, + \phi' - \lambda \\ &= \phi' - 10^\circ\ 55'\ 30'' \quad . \quad . \quad . \quad . \quad (5) \end{aligned}$$

$$\begin{aligned} LN - \lambda' &= LP_, + \phi' - \lambda' \\ &= \phi' - 29^\circ\ 5'\ 40'' \quad . \quad . \quad . \quad . \quad (6) \end{aligned}$$

Again, if $v_1\ v_2$ are the velocities of propagation in the direction $\theta\ \theta'$, we know that

$$v_1^2 + v_2^2 = a^2 + c^2 - (a^2 - c^2) \cos\theta \cos\theta'$$
$$v_1^2 - v_2^2 = (a^2 - c^2) \sin\theta' \sin\theta'$$

$a\ c$ having here their usual meanings of the greatest and least principal velocities. Hence

$$2v_1^2 = a^2 + c^2 - (a^2 - c^2) \cos(\theta + \theta')$$
$$2v_2^2 = a^2 + c^2 - (a^2 - c^2) \cos(\theta - \theta')$$

From these formulæ we can find $v_1\ v_2$, and thence the values of $\mu_1\ \mu_2$ for

$$\mu_1 = \frac{1}{v_1}$$

$$\mu_2 = \frac{1}{v_2}$$

Taking

$$\mu_a = 1{\cdot}68580$$

$$\mu_c = 1{\cdot}53013$$

$$c^2 = \frac{1}{\mu_a^2} = {\cdot}351876$$

$$a^2 = \frac{1}{\mu_c^2} = {\cdot}427117$$

TABLE XXV.—Comparison of FRESNEL's theory with experiment for the first prism in its first position.

	Outer sheet.				Inner sheet.			
	Excess of Experiment over Theory.	μ_2 Experiment.	μ_2 Theory.	$\theta - \theta'$.	$\theta + \epsilon'$.	μ_1 Theory.	μ_1 Experiment.	Excess of Experiment over Theory.
				° ′ ″	° ′ ″			
1	—·00006	1·68500	1·68506	7 21 20	18 21 20	1·68123	1·68119	—·00004
2	—·00002	1·68540	1·68542	5 13 20	18 20 0	1·68124	1·68119	—·00005
3	·00001	1·68568	1·68567	3 7 10	18 19 30	1·68124	1·68125	·00001
4	·00000	1·68578	1·68578	1 2 20	18 19 20	1·68125	1·68134	·00009
5	·00011	1·68589	1·68578	1 0 10	18 19 10	1·68125	1·68130	·00005
6	·00001	1·68568	1·68567	3 0 50	18 19 30	1·68124	1·68129	·00005
7	·00012	1·68558	1·68546	4 59 0	18 20 0	1·68124	1·68128	·00004
8	·00003	1·68517	1·68514	6 54 40	18 20 40	1·68123	1 68119	—·00004
9	—·00007	1·68467	1·68474	8 47 20	18 22 0	1·68122	1·68123	·00001
10	—·00004	1·68421	1·68425	10 36 40	18 24 0	1·68121	1·68116	—·00005
11	·00005	1·68376	1·68371	12 21 40	18 27 0	1·68118	1·68123	·00005
12	—·00010	1·68302	1·68312	14 1 30	18 32 30	1·68114	1·68113	—·00001
13	—·00025	1·68226	1·68251	15 32 40	18 43 40	1·68104	1·68114	·00010
14	—·00033	1·68190	1·68223	16 12 10	18 53 50	1·68096	1·68104	·00008
15	—·00046	1·68153	1·68199	16 45 20	19 9 40	1·68083	1·68106	·00023
16	—·00049	1·68131	1·68180	17 10 0	19 32 50	1·68063	1·68103	·00040
17	—·00044	1·68123	1·68167	17 26 50	20 2 50	1·68036	1·68073	·00037
18	—·00036	1·68122	1·68158	17 37 30	20 37 40	1·68005	1·68032	·00027
19	—·00030	1·68122	1·68152	17 44 50	21 15 40	1·67970	1·67996	·00026
20	—·00032	1·68114	1·68146	17 52 50	22 34 20	1·67893	1·67918	·00025
21	—·00018	1·68125	1·68143	17 56 50	23 52 40	1·67814	1·67841	·00027
22	—·00023	1·68117	1·68140	17 59 50	25 8 20	1·67731	1·67756	·00025
23	—·00016	1·68123	1·68139	18 1 30	26 19 30	1·67652	1·67674	·00022
24	—·00016	1·68122	1·68138	18 2 30	27 27 10	1·67574	1·67591	·00017
25	—·00003	1·68134	1·68137	18 3 30	28 29 30	1·67499	1·67519	·00020
26	—·00007	1·68130	1·68137	18 4 10	29 26 40	1·67428	1·67450	·00022
27	—·00016	1·68121	1·68137	18 4 40	30 18 20	1·67361	1·67388	·00027
28	—·00009	1·68127	1·68136	18 5 10	31 4 30	1·67301	1·67328	·00027
29	—·00015	1·68121	1·68136	18 5 20	31 45 40	1·67247	1·67274	·00027

TABLE XXVI.—Comparison of the results of FRESNEL'S theory with experiment for the first prism in its second position.

	Outer sheet.				Inner sheet.			
	Excess of Experiment over Theory.	μ_2 Experiment.	μ_2 Theory.	$\theta' \backsim \theta$.	$\theta + \theta'$.	μ_1 Theory.	μ_1 Experiment.	Excess of Experiment over Theory.
1	—·00008	1·68502	1·68510	7° 21′ 20″	18° 21′ 20″	1·68123	1·68117	—·00006
2	—·00008	1·68539	1·68547	5 13 20	18 20 0	1·68124	1·68122	—·00002
3	·00003	1·68570	1·68567	3 7 10	18 19 30	1·68124	1·68120	—·00004
4	—·00004	1·68574	1·68578	1 2 20	18 19 20	1·68125	1·68119	—·00006
5	—·00009	1·68569	1·68578	1 0 10	18 19 10	1·68125	1·68123	—·00002
6	—·00009	1·68560	1·68569	3 0 50	18 19 30	1·68124	1·68125	·00001
7	—·00008	1·68541	1·68549	4 59 0	18 20 0	1·68124	1·68125	·00001
8	—·00005	1·68517	1·68522	6 54 40	18 20 40	1·68123	1·68126	·00003
9	—·00009	1·68480	1·68489	8 9 22	18 21 32	1·68123	1·68126	·00003
10	—·00008	1·68439	1·68447	9 49 40	18 23 0	1·68121	1·68122	·00001
11	—·00015	1·68386	1·68401	11 26 3	18 25 13	1·68120	1·68122	·00002
12	—·00018	1·68333	1·68351	12 57 35	18 28 35	1·68116	1·68131	·00015
13	—·00021	1·68277	1·68298	14 22 48	18 34 28	1·68112	1·68127	·00015
14	—·00024	1·68223	1·68247	15 38 55	18 45 15	1·68103	1·68130	·00027
15	—·00019	1·68204	1·68223	16 11 55	18 53 55	1·68096	1·68140	·00044

The Tables XXV. and XXVI. give the results of theory for the first prism.

Let us take the outer sheet first.

In lines 1–12, Table XXV., and lines 1–10, Table XXVI., the agreement between theory and experiment is fairly close, but these series of experiments overlap and are in the immediate neighbourhood of an axis of the section.

From this point onwards, we see that the experimental curve lies entirely inside that given by FRESNEL'S theory, and the difference between the two increases as we approach the neighbourhood of the optic axes, line 16, Table XXV., 14 and 15 Table XXVI., while for the rest of Table XXV. the differences diminish.

The tables extend from about 8° on one side of the axis C, to nearly 16° on the other. This is obvious, for between the optic axes $\theta \backsim \theta'$, beyond them $\theta+\theta'$ is very approximately equal to twice N C.

Now let us consider the inner sheet, as before in lines 1–14, Table XXV., lines 1–11, Table XXVI., the agreement is fairly close; but these observations extend from about 6° on one side, to 7° on the other of the maximum radius vector of the section. As in the case of the outer sheet, the differences here increase as we approach the neighbourhood of the optic axes, reaching a maximum of ·00044, line 15, Table XXVI.

After passing the optic axes, the differences decrease, and for the rest of Table XXV. average about ·00024, and in this case the curve given by experiment lies outside that given by theory.

On examining the differences given in Table XXV. for the outer sheet, I noticed that theory and experiment agree closely up to the immediate neighbourhood of the optic axis, line 16; while in the neighbourhood of the optic axis, Table XXVI., line 15, the increase in the differences is much more gradual.

In fact, if we compare the experimental values of μ_1 in Table XXV., lines 11–14, and Table XXVI., 12–15, for about the same values of $\theta+\theta'$ we find considerable differences.

This led me to calculate the results of other sets of experiments covering the same ground, which had been already made.

The following table gives the results of one such set, taken about two months previously; during the interval the goniometer had been taken partially to pieces, and afterwards refocussed and set again.

It seems worth while to give the whole set of observations which cover nearly the same ground as Table II., and form therefore a test of the degree of accuracy of the results.

The measurements were made at angles of incidence, nearly, but not quite, coincident with those of Table II.

In order to compare results, so far as the wave E (inner sheet) is concerned, a correction by interpolation is necessary therefore.

The values of ϕ', it will be observed, differ by about 5′ from those in Table II.

In the table, the first column gives the values of ϕ'; the second the values of μ;

the third those of μ (corrected by interpolation for the wave E); and the fourth the values of μ from Table III.; while the last column gives the differences between the two sets.

Comparison of values of μ deduced from two independent series of observations at an interval of about two months.

TABLE XXVII.—Wave O.

	ϕ'.	μ.
	° ′ ″	
1	16 27 16	1·68131
2	17 32 15	1·68127
3	18 36 18	1·68128
4	19 39 22	1·68126
5	20 41 22	1·68129
6	21 42 8	1·68120
7	22 41 41	1·68128
8	23 39 55	1·68125
9	24 36 48	1·68116
10	25 32 5	1·68127
11	26 25 51	1·68116
12	27 17 37	1·68130
13	28 8 10	1·68116
14	29 32 20	1·68127
15	28 56 42	1·68106

TABLE XXVII. continued.—Wave E.

ϕ'.	μ.	Values of μ reduced to the values of ϕ' in Table II. by interpolation.	Values of μ from Table III. corresponding to the same values of ϕ' as in previous column.	Excess of the first series over the second.
° ′ ″				
29 19 59	1·68108	1·68115		
29 43 27	1·68066	1·68075	1·68073	·00002
30 6 16	1·68029	1·68038	1·68032	·00006
30 28 56	1·67991	1·68000	1·67996	·00004
31 12 9	1·67905	1·67914	1·67918	—·00004
31 53 13	1·67828	1·67837	1·67841	—·00004
32 32 10	1·67743	1·67752	1·67756	—·00004
33 8 28	1·67668	1·67678	1·67674	·00004
33 42 29	1·67590	1·67600	1·67591	·00009
34 13 53	1·67508	1·67518	1·67519	—·00001
34 42 22	1·67450	1·67461	1·67450	·00011
35 8 27	1·67368	1·67380	1·67388	—·00008

Taking the observations for the wave O first, we observe that they do not differ greatly from those given for the same values of ϕ' in Table II., except in the case of those which correspond to lines 12, 13, 14, of Table III., at which points the new values of μ_1 are greater than those given in Table III., and agree better with the observations in the neighbourhood of the other optic axis.

Another set of measurements in the same neighbourhood gave as values of μ_1

1·68124	1·68131
1·68123	1·68125

Hence, on the whole, it seems likely that the results of the observations given in Table III., lines 12, 13, 14, are too small, and that an error has been made in the measurement at that point, owing to the indistinctness of the images which nearly overlapped, so that we may fairly replace Table II., lines 12, 13, 14, by Table XXVII., lines 12, 13, 14, and instead of the results of Table XXV. referring to these we shall have

μ_1 Theory.	μ_1 Experiment.	Difference.
1·68114	1·68130	·00016
1·68104	1·68116	·00012
1·68096	1·68127	·00031

This modification renders the results similar in the neighbourhood of the two optic axes.

Let us now consider the results given in the latter part of the table for the wave E.

The two sets of observations there recorded were made under entirely different circumstances. The telescope and collimator of the goniometer were removed in the interval between them, and both reset and refocussed; and yet in but one case is the difference between the results of the two observations as great as ·0001, while the average difference, irrespective of sign, is about ·00005.

This comparison forms a strong test of the accuracy of the experiments, and the close agreement of the results seems to show that the error in them is at any rate not greater than ·00005.

It would appear that more trustworthy results might have been obtained by combining the series of results registered in this last table with those discussed throughout.

But experiments covering the whole ground embraced in the series we have chosen had not been made at the same time previously.

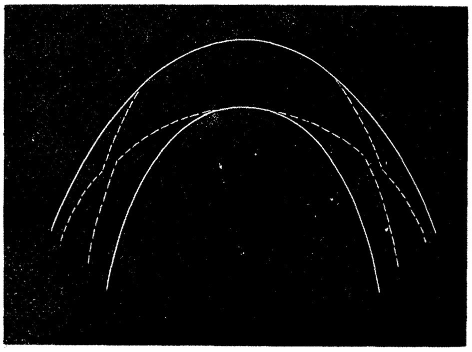
Moreover, we must recollect that each value of ϕ and $D+i$, given in Tables I., IV., &c., is the mean of three for the first prism and of two for the second.

And, again, this comparison assures us that the difference between the results of experiment and theory amounting, as it does in some cases, to ·0005 in the value of μ, is far too great to have arisen from experimental errors or errors of observation.

It seems, then, that assuming the position of the plane of the prism, and the values of the principal axes to be accurately determined, we may assert that in a central section of the wave surface inclined at a small angle to the plane of the optic axes, there is considerable difference between FRESNEL'S theory and experiment: that the differences between the two are most marked in the neighbourhood of the optic axes, and amount there to ·0005 about. For the outer sheet of the surface (except just in the neighbourhood of the principal axes), the theoretical values of the radius vector are uniformly greater than the experimental, while for the inner sheet the reverse is the case.

In fact, the curves of section of the surface, as given by experiment, approach more nearly to one another than those of the surface of wave slowness on FRESNEL'S theory.

Fig. 8.



In the above figure, fig. 8, the dotted line gives the result of experiments, while the strong line gives the form of the section on FRESNEL'S theory.

To compare the results with those given by Lord RAYLEIGH'S theory, we have the equation to the surface, a, b, c being the principal velocities,

$$\frac{l^2}{\frac{V^2}{a^2}-1}+\frac{m^2}{\frac{V^2}{b^2}-1}+\frac{n^2}{\frac{V^2}{c^2}-1}=0$$

Put $V=\frac{1}{r}$, r is a radius vector of the surface of wave slowness

$$\frac{a^2l^2}{1-a^2r^2}+\frac{b^2m^2}{1-b^2r^2}+\frac{c^2n^2}{1-c^2r^2}=0$$

Let us suppose as a first approximation that $m=0$, *i.e.*, that the plane of the prism is coincident with a principal plane of the crystal,

$$a^2l^2(1-c^2r^2)+c^2n^2(1-a^2r^2)=0$$

$$r^2a^2c^2=a^2l^2+{}^2cn^2$$

$$r^2=\frac{l^2}{c^2}+\frac{n^2}{a^2}$$

Also $l=\cos\theta$, $n=\sin\theta$

θ being the angle between r and the maximum radius vector

$$r^2=\frac{1}{c^2}-\left(\frac{1}{c^2}-\frac{1}{a^2}\right)\sin^2\theta$$

$$\frac{1}{c^2}=2\cdot84192$$

$$\frac{1}{c^2}-\frac{1}{a^2}=\cdot50062.$$

From these and the known values of θ, μ or r can be calculated. The table gives the result.

Comparison between Lord RAYLEIGH's Theory and Experiment.

θ.	μ Theory.	μ Experiment.	Excess of Theory over Experiment.
1° 31′ 10″	1·68570	1·68568	·00002
8 25 33	1·68261	1·68190	·00071
11 6 40	1·68028	1·67918	·00110
14 9 48	1·67688	1·67519	·00169
15 48 28	1·67476	1·67274	·00202

The differences between theory and experiment are so marked that it seems unnecessary to calculate the values of μ for smaller intervals in the values of θ.

And though some of the apparent difference may be due to the error made in assuming the principal plane of the prism to coincide with one of the crystal, that cannot account for the whole; for we have seen that in FRESNEL's surface the error made by the same assumption appears only in the fourth place of decimals, in the value of the refractive index, while the differences between Lord RAYLEIGH's theory and experiment show themselves in the third place, and tend to increase with θ.

Thus it seems clear that Lord RAYLEIGH's theory will not account for the phenomena of double refraction in arragonite. This result agrees with that arrived at by Professor STOKES for Iceland spar.

Section VII.—*Effect of Varying Constants of Theory and Position of Plane of Prism with reference to Axes of Crystal.*

It remains now to discuss the effects of variations in the values of the axes a, b, c.

The values of c and b are determined directly from observation, and in the neighbourhood of these axes the agreement is most close. Any alteration in them would affect especially that part of the section which lies between the optic axes, and for which the differences between theory and experiment are least; let us consider, therefore, a variation in the value of a.

Such a variation would produce an alteration in the angle between the optic axes, and therefore in the values of θ, θ' for any wave normal.

If, however, we take a wave normal at some distance from the optic axes, the values of θ, θ' will be altered by nearly equal amounts in opposite directions, and $\theta+\theta'$ will be nearly constant.

Let us then find the value for a in order that the experimental value in line 29 of Table XXV. may agree with theory.

We have

$$2v_1^2=a^2+c^2-(a^2-c^2)\cos(\theta+\theta')$$

$$=a^2\{1-\cos(\theta+\theta')\}$$

$$+c^2\{1+\cos(\theta+\theta')\}$$

$$\therefore a^2=\frac{v_1^2-c^2\cos^2\dfrac{\theta+\theta}{2}}{\sin^2\dfrac{\theta+\theta'}{2}}$$

$$v_1=\frac{1}{\mu_1}$$

$$\mu_1=1{\cdot}67274$$

$$\theta+\theta'=31^\circ\ 45'\ 40''$$

$$c^2={\cdot}351876$$

From these we get

$$a^2={\cdot}447553$$

as against ·427117, and the value of μ_c is

$$\mu_c=1{\cdot}49478$$

instead of 1·53013

A reference to the tables for the second prism shows at once the impossibility of such a change.

It is true that any decrease in μ_c would produce a decrease in the angle between the optic axes. This would increase θ' and decrease θ by nearly equal amounts, and on the whole produce a small decrease in $\theta+\theta'$.

This change would decrease v_1 and therefore increase μ_1, and hence to produce the required change in μ_1—*i.e.*, to make μ_1 equal to the experimental value in line 29—we must take a value for μ_c larger, but only slightly larger, than that given above.

There remains now for discussion the effect of a variation in the position of the plane of the prism with reference to the crystallographic axes.

We shall consider this by discussing separately the effect of a rotation about each of three lines through the centre approximately at right angles. These lines are

(1.) The normal to the principal plane of the prism; (2.) The intersection of this plane with the plane B C B′; (3.) The intersection of the principal plane with the plane A C A′.

(1.) Rotation about a normal to the principal plane of the prism.

This cannot produce the required effect, for though we can thus decrease the differences on one side of the maximum radius vector of the section, we increase those on the other side by about the same amounts.

(2.) Rotation about the intersection with B C B′.

This is of no use, for the effects produced on opposite sides of the maximum radius vector are again of an opposite kind.

(3.) There remains, therefore, only the rotation about the line of intersection with the plane C A C′, or a variation in the angle χ. This change is to be such that the sections of the outer and inner sheets may be brought into closer proximity; χ must therefore be decreased.

Now on referring to the tables of the experimental results, it will be seen that none of those in Table III., lines 1–11, Table VI., lines 16–29, and Table IX., lines 1–16, differ greatly from 1·68125.

If it were then permissible to neglect the angle χ, this would be the value of μ_b, and the other section would be on FRESNEL'S theory an ellipse of μ_a, μ_c.

In the abstract published in the 'Proceedings of the Royal Society,' No. 188, 1878, this has been done, and a limit assigned to the error thus introduced.

Professor STOKES has since pointed out that near the optic axes the limit is considerably too small, and thus led me to the accurate calculation of the theoretical values of μ_1, μ_2 given above.

The effect of a decrease in the angle χ may, therefore, be estimated by referring to the calculations for the case in which we put $\chi=0$.

The accompanying table gives the results. θ being the angle between the wave normal and the major axis of the ellipse, which is given by

$$\theta=\phi'\backsim\theta_0$$

where θ_0 is the angle between the normal to the face of the prism and major axis, and is

$$=20^\circ\ 0'\ 40''\quad \text{[Section V. (2)]}.$$

Table XXVIII.

Tables from which experimental result is taken.	θ.	r or μ from Theory.	r or μ from Experiment.	Excess of Experiment over Theory.
	° ′ ″			
XI.	8 25 38	1·68193	1·68204	
”	8 4 45	1·68224	1·68223	− 1
”	7 21 11	1·68284	1·68277	− 7
”	6 35 31	1·68342	1·68333	− 9
”	5 47 55	1·68396	1·68386	−10
”	4 58 26	1·68444	1·68439	− 5
”	4 7 22	1·68486	1·68480	− 6
V.	3 42 55	1·68504	1·68500	− 4
XI.	3 14 28	1·68522	1·68517	− 5
V.	2 38 14	1·68542	1·68540	− 2
XI.	2 20 10	1·68550	1·68541	− 9
V.	1 34 26	1·68566	1·68570	+ 4
XI.	1 23 23	1·68570	1·68560	−10
V.	0 31 31	1·68578	1·68578	− 0
XI.	0 27 12	1·68578	1·68569	− 9
V.	0 30 15	1·68578	1·68589	+11
XI.	0 31 18	1·68577	1·68574	− 3
XI.	1 31 3	1·68567	1·68570	+ 3
V.	1 31 10	1·68567	1·68568	+ 1
”	2 30 43	1·68544	1·68560	..
XI.	2 31 48	1·68544	1·68539	− 5
V.	3 29 17	1·68515	1·68517	+ 2
XI.	3 33 40	1·68510	1·68502	− 8
V.	4 26 32	1·68472	1·68467	− 5
”	5 22 18	1·68422	1·68421	− 1
”	6 16 31	1·68367	1·68376	+ 9
”	7 9 25	1·68300	1·68302	+ 2
”	8 0 38	1·68230	1·68226	− 4
”	8 25 33	1·68194	1·68190	− 4
”	8 50 1	1·68156	1·68153	− 3
II.	9 14 14	1·68118	1·68131	..
”	9 37 36	1·68077	1·68073	− 4
”	10 0 40	1·68037	1·68032	− 5
”	10 23 7	1·67997	1·67996	− 1
”	11 6 40	1·67914	1·67918	+ 4
”	11 47 59	1·67831	1·67841	+10
”	12 27 8	1·67748	1·67756	+ 8
”	13 3 53	1·67666	1·67674	+ 8
”	13 38 11	1·67587	1·67591	+ 4
”	14 9 48	1·67511	1·67519	+ 8
”	14 38 44	1·67439	1·67450	+11
”	15 4 51	1·67373	1·67388	+15
”	15 28 8	1·67310	1·67328	+18
”	15 48 28	1·67250	1·67274	+24

For about 10° on either side of the major axis (the passage across which is indicated by the dark line in the table) the results of theory and experiment agree fairly well.

The average difference irrespective of sign is (if we neglect the values for $\theta=2° 30' 43''$ and for $\theta=9° 14' 14''$, which gives a point on the circle of radius 1·68125) ·000047, and only in two cases does the difference amount to ·0001.

But when we come to values for θ greater than 11° we see that the experimental value of μ is, as before, always greater than the theoretical, and that the difference increases with θ, and at last becomes nearly the same as in Table XXV.

Thus this change does not produce agreement between the two.

We may, however, consider shortly what alteration in μ_c would in this case bring theory and experiment more closely into agreement.

We have

$$\frac{1}{\mu^2}=\frac{\cos^2\theta}{\mu_a^2}+\frac{\sin^2\theta}{\mu_c^2}$$

$$\therefore \delta\mu=\frac{\mu^3}{\mu_c^3}\sin^2\theta\delta\mu_c$$

Taking the last values on Table XXVIII.

$$\delta\mu=\cdot00024$$

$$\theta=15° 48'$$

$$\mu=1\cdot67274$$

$$\mu_c=1\cdot53013$$

Whence

$$\delta\mu_c=\cdot00248$$

so that the minor axis of the ellipse, which having the same major axis as the experimental curve would pass through the extreme experimental point, is

$$\mu_c=1\cdot53261$$

instead of 1·53013

And on referring to Table XXI., which gives the values of μ_c very approximately, we see that this value is quite out of the question.

Thus a decrease in the angle χ will not produce the required effect.

Neither will an increase. For the first part of the inner sheet in Table III., and throughout Table IX., we must have the theoretical value of μ_1 and therefore of $\theta+\theta'$ nearly constant; and this is clearly impossible along a plane section inclined at a finite angle to the plane of the optic axes.

Thus a change in χ will not render theory and experiment consistent.

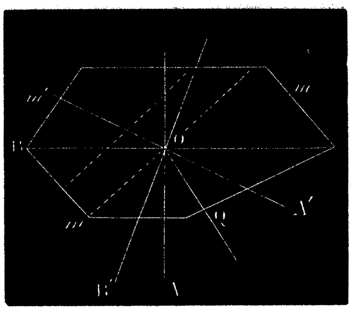
Thus no change in the position of the plane of the prism will bring FRESNEL'S theory into agreement with experiment.

Section VIII.—*The Theoretical Investigations for the Second Prism on* FRESNEL'S *Theory with Comparison with Experiment.*

Our next step is the theoretical investigation for the second prism.

Now experiment has shown that this prism lies in the twin crystal; we must therefore find the axes of this crystal.

Fig. 9.



Let the figure represent a section of the crystal by a plane perpendicular to the axis O C.

The twinning takes place about an axis perpendicular to m', and consists of a rotation through 180° about that axis.

Its effect, therefore, is to bring O B, fig. 9, into the position O B′, where

$$\text{angle } m\text{OB} = m\text{OB}'$$

that is, to turn the axes in the direction B to A, through an angle,

$$= 2(m\text{OB}) = 2\left(\frac{\pi}{2} - \text{AO}m\right)$$

$$= \pi - 2\mu = 180° - 116° \ 56'$$

$$= 63° \ 4'$$

and if Q is the point where the perpendicular on the face Q of the second prism meets the sphere, we know that

$$\text{AQ} = 45° \ 30' \quad \text{[Section IV. (1)]}.$$

Hence since AA′$=63° \ 4'$ and Q lies in plane A O B,

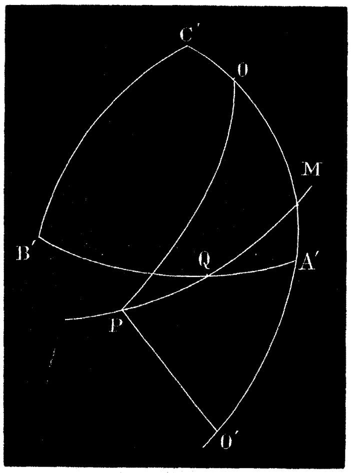
$$\text{A}'\text{OQ} = 17° \ 34'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now let A′ B′ C′, fig. 10, be the points in which the new axes cut the sphere, and let

P Q M be the plane of the prism; P being any point in the plane; Q the normal to the face, and M the point in which the plane cuts B′ C′, O, O′ the optic axes.

Then if P is the pole of any wave in the crystal, $PQ=\phi'$.

Fig. 10.



Let $PO=\theta$, $PO'=\theta'$, we shall require the values of θ, θ'.

Now in triangle A′ M Q

$$\text{angle } Q=4^\circ\ 37'\ 20'' \quad \text{[Section IV. (4)].}$$

$$A'Q=17^\circ\ 34'$$

$$\cot MQ=\cos Q \cot A'Q$$

whence

$$MQ=17^\circ\ 37'\ 14'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let $A'MQ=\chi$.

Then from triangle A′ M Q,

$$\cos M=\cos QA' \sin Q.$$

whence

$$\chi=85^\circ\ 35'\ 40'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now,

$$\cos\theta'=\cos PM\cos MO'+\sin PM\sin MO'\cos\chi$$

$$=\cos MO'(\cos PM+\sin PM\tan MO'\cos\chi)$$

put

$$\tan MO'\cos\chi=\tan\lambda'$$

$$\cos\theta'=\frac{\cos MO'\cos(PM-\lambda')}{\cos\lambda'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Similarly

$$\cos\theta=\frac{\cos \text{MO} \cos (\text{PM}+\lambda)}{\cos\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where

$$\tan\lambda=\tan \text{MO} \cos\chi$$

Now

$$\text{A}'\text{O}=\text{A}'\text{O}'=90^\circ-9^\circ\ 4'\ 58'' \quad [\text{Section V. (6)}].$$
$$=80^\circ\ 55'\ 2''$$

From triangle A′ M Q

$$\sin \text{A}'\text{M}=\sin \text{MQ} \sin \text{Q}$$

Whence

$$\text{A}'\text{M}=1^\circ\ 23'\ 52'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\therefore \text{MO}=79^\circ\ 31'\ 10''. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\text{MO}'=82^\circ\ 18'\ 54''. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Hence

$$\lambda=22^\circ\ 33'\ 2''. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\lambda'=29^\circ\ 39'\ 5'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\text{PM}+\lambda=\phi'+\text{QM}+\lambda$$
$$=\phi'+40^\circ\ 10'\ 16'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\text{PM}-\lambda'=\phi'+\text{QM}-\lambda'$$
$$=\phi'-12^\circ\ 1'\ 51'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Substituting these values of λ, MO, PM$+\lambda$, &c. (the values of ϕ' being taken from Tables XIV., XVII., and XX.), in equations (4), (5), we can get a series of values for θ θ'.

And as before [Section VI.], remembering that θ' here is $\pi-\theta'$ in Section VI.,

$$2v_1^2=a^2+c^2+(a^2-c^2)\cos(\theta-\theta')$$
$$2v_2^2=a^2+c^2+(a^2-c^2)\cos(\theta+\theta')$$

Thus we obtain a series of values for v_1, v_2, and hence for μ_1, μ_2, given in Table XXIX.

To obtain the distance of P from the minimum radius vector, we have to add to ϕ' the angle M Q$=17^\circ\ 37'\ 14''$ [Section VIII. (2)].

Table XXIX.—Comparison of Fresnel's theory and experiment for second prism.

	Outer sheet.					Inner sheet.				
	φ′ from Tables XVII. and XXIII.	Excess of Experiment over Theory.	μ_2 Experiment.	μ_2 Theory.	$\theta+\theta'$.	$\theta-\theta'$.	μ_1 Theory.	μ_1 Experiment.	Excess of Experiment over Theory.	φ′ from Tables XIV. and XX.
	° ′ ″				° ′ ″	° ′ ″				° ′ ″
1	12 16 53	−·00024	1·68218	1·68242	164 15 15	1 17 0	1·53014	1·53014	..	8 21 52
2	12 48 1	−·00026	1·68220	1·68246	164 20 17					
3	13 21 57	−·00010	1·68240	1·68250	164 25 43					
4	13 57 54	−·00027	1·68227	1·68254	164 31 42					
5	14 36 29	−·00028	1·68231	1·68259	164 38 7			1·53010		
6	15 17 33	−·00009	1·68254	1·68263	164 45 9	1 57 45	1·53016	1·53003	−·00013	11 48 45
7	16 0 36	−·00013	1·68256	1·68269	164 52 32			1·53010		
8	16 45 43	−·00016	1·68256	1·68272	165 0 34					
9	17 32 54	−·00015	1·68264	1·68279	165 8 56					
10	18 21 59	−·00015	1·68269	1·68286	165 18 10					
11	19 13 4	−·00003	1·68289	1·68292	165 27 46	2 35 0	1·53019	1·53013	−·00006	16 16 9
12	20 5 38	−·00003	1·68295	1·68298	165 37 52					
13	20 35 20	+·00010	1·68312	1·68302	165 43 32					
14	21 36 0	+·00020	1·68330	1·68310	165 55 42					
15	22 35 30	+·00019	1·68336	1·68317	166 7 38					
16	23 33 27	+·00022	1·68347	1·68325	166 19 42	4 2 10	1·53028	1·53028	..	26 5 11
17	24 30 19	+·00025	1·68357	1·68332	166 31 32					
18	25 25 35	+·00024	1·68363	1·68339	166 43 48					
19	26 19 10	+·00032	1·68378	1·68346	166 55 6					
20	27 11 11	+·00028	1·68381	1·68353	167 6 38					
21	28 1 21	+·00031	1·68391	1·68360	167 17 56	4 43 34	1·53035	1·53036	+·00001	31 7 42
22	28 49 40	+·00036	1·68400	1·68366	167 28 54					
23	29 36 10	+·00029	1·68401	1·68372	167 39 32					
24	30 20 28	+·00035	1·68413	1·68378	167 50 0					
25	31 2 40	+·00038	1·68421	1·68383	167 59 56					
26	31 42 34	+·00048	1·68437	1·68389	167 9 29	5 24 56	1·53043	1·53045	+·00002	35 20 36

Considering first the inner sheet, we see that theory and experiment agree closely.

The greatest difference is ·00013; and in this case, by referring to the adjacent experimental values, it is clear that the value 1·53003 is too small.

We must remember, however, that the investigations are concerned with that sheet of the surface of wave slowness of which the section by the plane, $x\ y$, is a circle, and that the section considered is inclined at an angle less than 5° to this plane; so that the form of the curve of section differs but little from a circle, and we may therefore reasonably suppose that any theory in which one of the principal sections of the surface of wave slowness is a circle, would also agree fairly closely with our results.

We therefore proceed to discuss the results for the outer sheet given in the table just preceding.

And here we see at once that there is considerable difference between theory and experiment.

Moreover, from $\phi'=12° 16' 53''$ to $\phi'=20° 5' 38''$, the theoretical values of μ are uniformly greater than the experimental; while from $\phi'=20° 35' 20''$ to $\phi'=31° 42' 34''$ the reverse is the case.

The difference between the two increases on the whole uniformly, and in the last case is as great as ·0005.

For the values $\phi'=19° 13' 4''$ and $\phi'=20° 5' 38''$, theory and experiment agree closely.

To find the angle between any radius vector and the minimum one of the curve, we have to add to the value of ϕ' the angle M Q, which is 17° 37′ 14″.

Let us call the angle Q O P, θ.

Then we see that for the angles $\theta=0$, $\theta=37°$ (about), and $\theta=90°$, the curves given by theory and experiment agree.

From $\theta=29° 54' 7''$ to $\theta=37°$ (about), the theoretical curve lies outside the experimental.

From $\theta=37°$ (about) to $\theta=49° 19' 48''$ the reverse is the case.

In the neighbourhood of the axes the two must agree more closely.

These conditions would all be satisfied by a curve which agrees with FRESNEL'S section at the extremity of the minimum radius vector; lies inside it for about 40°; then cuts FRESNEL'S curve again, and lies outside for the rest of the quadrant, agreeing again at the extremity of the maximum radius vector.

At first sight there seems some discrepancy between this result and that arrived at already in Table XXV. for the first prism.

But closer examination shows that the two results are corroborative.

For, in the first place, the observations there tabulated apply to the neighbourhood of the major axis of the elliptic section.

For about 6° on either side of this principal axis, the differences between theory and experiment are sometimes positive, sometimes negative; that is, between these limits FRESNEL'S section satisfies the results of experiment; but in the last eighteen observations there recorded, the experimental value of μ_1 is always greater than the theoretical.

Thus the experimental results are represented by a curve, which near the major axis agrees with FRESNEL'S section, but which, as we go from 6° to 16° away from that axis, differs from that section, and lies outside the section.

This result, viz.: that on going away from the maximum radius vector the curve given by experiment should lie outside that given by theory, is exactly that to which we have been led by the work for the second prism.

The cutting of the crystal would not allow of observations being taken near the minor axis of the ellipse, in the case of the first prism.

Of course, it has here been assumed that the sections of the surface of wave slowness, by the principal planes, are curves of the same kind; or rather the results of experiment lead to that as a fact.

Lord RAYLEIGH'S theory needs no further discussion here, for his curve being the inverse of an ellipse, always lies outside the ellipse with the same axes, and therefore cannot agree with experiment.

Section IX.—*Possibility of an Error in the Position of the Crystallographic Axes discussed.—No Section of the Surface of Wave Slowness in* FRESNEL'S *Theory can agree with the experimental Results.*

But the possibility remains that there is an error in the determination of the position of the faces of the prisms, especially in the second case, with respect to the crystallographic axes.

The probability of any error is small, as the observations were repeated on several different occasions with nearly coincident results.

To make certain, however, let us solve the inverse problem. Having given the two values, μ_1 μ_2, of the radii vectores of the surface of wave slowness drawn in the same direction l m n in the crystal to find l, m, n.

We have as the equation to find v on FRESNEL'S theory

$$\frac{l^2}{v^2-a^2}+\frac{m^2}{v^2-b^2}+\frac{n^2}{v^2-c^2}=0$$

Whence

$$v^4-v^2\{l^2(b^2+c^2)+m^2(c^2+a^2)+n^2(a^2+b^2)\}+l^2b^2c^2+m^2c^2a^2+n^2a^2b^2=0$$

$\therefore$ if v_1 v_2 are the roots of this equation, that is, the velocities for the same wave front

$$v_1^2v_2^2=l^2b^2c^2+m^2c^2a^2+n^2a^2b^2$$
$$v_1^2+v_2^2=l^2(b^2+c^2)+m^2(c^2+a^2)+n^2(a^2+b^2)$$

but $n^2=1-m^2-l^2$

Whence we get

$$b^2l^2(c^2-a^2)+a^2m^2(c^2-b^2)=v_1^2v_2^2-a^2b^2$$
$$l^2(c^2-a^2)+m^2(c^2-b^2)=v_1^2+v_2^2-a^2-b^2$$

Whence

$$l^2(a^2-c^2)(a^2-b^2)=(a^2-v_1^2)(a^2-v_2^2)$$
$$m^2(b^2-c^2)(b^2-a^2)=(b^2-v_1^2)(b^2-v_2^2)$$
$$n^2(c^2-a^2)(c^2-b^2)=(c^2-v_1^2)(c^2-v_2^2)$$

From the tables for the second prism, we see that for the inner sheet for

$$\phi'=28°\ 11'\ 21''\ \frac{1}{v_1}=1\cdot53021$$

For the outer sheet,

$$\phi'=28°\ 1'\ 21''\ \frac{1}{v_2}=1\cdot68391$$

Since the value of v_1 changes very slowly, we may reasonably treat these two values for ϕ' as coincident, and substitute in the above formulæ.

We find

$$\cos^{-1} n_1=49°\ 35'$$
$$\cos^{-1} m_1=40°\ 27'$$

Again, for inner sheet for

$$\phi'=38°\ 2'\ 41''\ \frac{1}{v_1}=1\cdot53037$$

For outer sheet

$$\phi'=31°\ 42'\ 34''\ \frac{1}{v_2}=1\cdot68437$$

Treating these two as coincident, we find

$$\cos^{-1} n_2=55°\ 44'$$
$$\cos^{-1} m_2=34°\ 23'$$

For inner sheet for

$$\phi'=26°\ 5'\ 11''\ \frac{1}{v_1}=1\cdot53028$$

For outer sheet

$$\phi'=26°\ 19'\ 10''\ \frac{1}{v_2}=1\cdot68378$$

Whence

$$\cos^{-1} n_3=47°\ 56'$$
$$\cos^{-1} m_3=42°\ 8'$$

We may justify the treating of the two values of ϕ' as the same, by the consideration that μ_1 varies at the rate of ·00001 (about) for 1°, and therefore the difference for 20′ is inappreciable.

Now the angle between the radii vectores in the first and second case cannot be less than

$$\cos^{-1} n_2 - \cos^{-1} n_1$$

i.e.,

$$\text{not less than } 55^\circ\ 44' - 49^\circ\ 35'$$
$$\text{or } 6^\circ\ 9'$$

But the angle between these radii is the difference between the two corresponding values of ϕ', or

$$= 31^\circ\ 42'\ 34'' - 28^\circ\ 1'\ 21''$$
$$= 3^\circ\ 41'\ 13''$$

Again, taking the second and third, the angle deduced from theory is not less than

$$\cos^{-1} n_2 - \cos^{-1} n_3$$
$$\text{or } 55^\circ\ 44' - 47^\circ\ 56'$$
$$\text{or } 7^\circ\ 48'$$

but from experiment it is

$$31^\circ\ 42'\ 34'' - 26^\circ\ 19'\ 10''$$
$$= 5^\circ\ 23'\ 24''$$

In both these cases the angle, as given by theory, must be very much larger than that given by experiment.

Thus, in order that the three pairs of values of $\mu_1\ \mu_2$ may correspond to three radii of the surface of wave slowness on FRESNEL'S theory, the angles between them must be half as large again as they are found to be.

Hence no section of the surface of wave slowness on FRESNEL'S theory can agree with the results of the experiments. We may remark, in addition, that the differences between consecutive values of μ for the outer sheet given by experiment are about double those given by theory.

After this investigation it seems needless to discuss possible alterations in the position of the plane of the second prism, with a view to bringing the results of experiment more nearly into accordance with theory. While as regards the constants a, b, c, it has already been seen that they have received their most probable values.

Thus our results, so far as they go, point to the fact that FRESNEL'S construction does not represent Nature, and that some other theory must be sought to explain the phenomena of double refraction.

And as regards the rival theories already proposed, Lord RAYLEIGH'S, we have seen, is even more than FRESNEL'S at variance with observed appearances, while I can hardly think that sufficient data have yet been collected to repay the labour of comparison with the theories of GREEN and CAUCHY, for on account of the undetermined constants it would be necessary to force theory and experiment to agree in so many points that they must, perforce, almost agree exactly along the small arcs investigated.

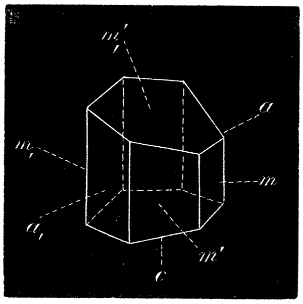
PART II.

Section I.*—*Description of a Second Crystal and Results of Experiments.*

Before asking permission to lay the results obtained in the previous part of the paper before the Society, I thought it would be better, if possible, to repeat the investigation with a second piece of the crystal. With some trouble I obtained a suitable piece last February from A. HILGER, of Tottenham Court Road.

Like the former piece, it was an hexagonal prism in shape, but of considerably smaller cross-section.

Fig. 1.

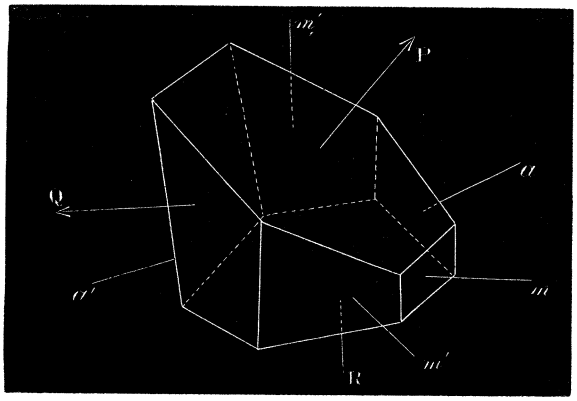


The axis of the prism was parallel to the axis of c, the base being perpendicular to the axis, while the top was broken off obliquely.

The planes $m_{,}$ m' gave the best reflexions, and were therefore chosen to determine the position of artificial faces.

My chief aim in having the prism cut was to get as long a continuous arc on the wave surface as possible. To attain this I had a face R, fig. 2, polished as nearly coincident with c as possible.

Fig. 2.



The oblique end was cut (P, fig. 2) so as to be inclined at about 37° to R, the edge of the prism thus formed between P and R being nearly parallel to the intersection of m and c.

* N.B. The two Parts being quite distinct, the numbering of the Sections and Figures is commenced afresh with Part II.

The face $m_{,}$ was cut so as to be inclined at about 35° 20′ to P, the line of inter section of this face (Q, fig. 2) and P being also nearly parallel to that of P and R.

So that the principal planes of the prisms formed by P and R, P and Q respectively, were nearly coincident, and as my observations extended almost from perpendicular incidence on Q to perpendicular incidence on R: they embraced an arc of over 70°.

The instrument used and the methods adopted for levelling, reading, &c., have been fully described in the previous part of the paper.

The results of the experiments are contained in Tables I., II., III., and IV.

Tables I. and II. refer to the prism R P.

ϕ is the angle made by the wave normal in air with the normal to P.

ϕ' the corresponding angle in the crystal.

D is the observed deviation.

i the angle of prism.

$D+i$ is given, for it occurred in the calculation, and was just as easily found as D.

The value taken for i is

$$i=37° \; 2' \; 56''$$

This value is the mean of twelve observations, none of which differed from the mean by 10″.

ϕ' is found from the formulæ $\phi'+\psi'=i$

$$\text{and } \tan\frac{\phi'-\psi'}{2}=\tan\frac{i}{2}\cot\frac{\phi+\psi}{2}\tan\frac{\phi-\psi}{2}$$

which has already been proved (Part I., Section I.).

In Table I. ϕ is the same for both outer and inner sheet, being the angle of incidence.

In Table II. ϕ has different values for the two sheets, being the angle of emergence, for as already explained, on passing through the position of minimum deviation, the prism was reversed so that the face of incidence became that of emergence.

Throughout the work μ_1 refers to the inner, μ_2 to the outer sheet.

The values of $\delta\phi'$, $\delta\mu$ are calculated as before on the assumption of errors of 10″ in 2ϕ and $D+i$, the observed quantities taken; these errors being combined so as to produce the greatest possible effect in the result.

Tables III. and IV. refer to P Q, for which $i=35° \; 18' \; 50''$.

The symbols have the same meaning, but while in I. and II. the wave normal falls on the same side of the normal to P as the axis O C, in III. and IV. it is on the other side of O P.

In III. ϕ is the angle of emergence, and therefore different for the inner and outer sheets; in IV. it is the angle of incidence, and the same for both.

Each observation was repeated twice on different occasions; the results of two measurements rarely differed by 20′.

Table I.

Outer sheet.					ϕ.	Inner sheet.					
$\delta\mu_2$.	μ_2.	$\delta\phi'$.	ϕ'.	$D+i$.		$D+i$.	ϕ'.	$\delta\phi'$.	μ_1.	$\delta\mu_1$.	
		″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″			
			35 59 41	83 53 56	82 7 10	83 42 26	36 6 16		1·68102		1
	1·68559		35 45 55	82 17 1	80 7 10	82 5 36	35 52 30		1·68112		2
·00009	1·68558	9	35 29 24	80 44 41	78 7 10	80 33 21	35 35 59	9	1·68106	·00009	3
	1·68553		35 10 1	79 17 31	76 7 10	79 6 26	35 16 20		1·68115		4
	1·68546		34 47 54	77 54 51	74 7 10	77 44 6	34 53 57		1·68114		5
	1·68539		34 22 47	76 37 16	72 7 10	76 26 36	34 28 43		1·68115		6
·00007	1·68532		33 55 3	75 24 6	70 7 10	75 13 26	34 0 54		1·68106		7
	1·68526	8	33 24 38	74 15 31	68 7 10	74 4 51	33 30 24	8	1·68099	·00007	8
	1·68519		32 51 40	73 11 16	66 7 19	73 0 36	32 57 21		1·68089		9
	1·68505		32 16 18	72 11 11	64 7 10	72 0 31	32 21 52		1·68074		10
	1·68514		31 38 14	71 15 51	62 7 10	71 4 51	31 43 51		1·68069		11
	1·68497		30 58 13	70 23 51	60 7 10	70 12 26	31 3 56		1·68030		12
·00006	1·68489	8	30 15 49	69 36 6	58 7 10	69 24 31	30 21 28	8	1·68016	·00006	13
	1·68485		29 31 16	68 52 16	56 7 10	68 39 31	29 37 20		1·67962		14
	1·68465		28 44 50	68 11 51	54 7 10	67 58 26	28 51 1		1·67917		15
	1·68473		27 56 13	67 35 36	52 7 10	67 20 41	28 2 53		1·67859		16
	1·68463		27 5 54	67 2 36	50 7 10	66 46 36	27 12 48		1·67806		17
·00004	1·68454	7	26 13 50	66 33 6	48 7 10	66 15 16	26 21 14	7	1·67721	·00004	18
	1·68457		25 19 58	66 7 16	46 7 10	65 47 16	25 27 58		1·67634		19
	1·68457		24 24 34	65 44 56	44 7 10	65 22 21	24 33 11		1·67532		20
	1·68459		23 27 40	65 26 1	42 7 10	65 0 31	23 36 55		1·67421		21
	1·68451		22 29 27	65 10 11	40 7 10	64 41 51	22 39 13		1·67303		22
·00002	1·68448	6	21 29 52	64 57 56	38 7 10	64 25 56	21 40 18	6	1·67161	·00002	23
	1·68441		20 29 5	64 49 6	36 7 10	64 13 16	20 40 3		1·67015		24
	1·68446		19 27 3	64 44 6	34 7 10	64 3 36	19 38 38		1·66854		25
	1·68461		18 23 54	64 42 56	32 7 10	63 57 26	18 36 0		1·66694		26
	1·68446		17 19 54	64 45 1	30 7 10	63 54 1	17 32 26		1·66502		27
	1·68448	6	16 14 53	64 51 36	28 7 10	63 54 6	16 27 50	6	1·66300		28

TABLE II.

Outer sheet.						Inner sheet.						
ϕ.	$\delta\mu_2$.	μ_2.	$\delta\phi'$.	ϕ'.	$D+i$.	ϕ.	$D+i$.	ϕ'.	$\delta\phi'$.	μ_1.	$\delta\mu_1$.	
° ′ ″			″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″			
32 49 16		1·68433		18 46 18	64 42 26	32 3 51	63 57 1	18 34 20		1·66678		1
30 50 31		1·68458		17 43 17	64 43 41	30 0 46	63 53 56	17 29 1		1·66490		2
28 55 16		1·68446		16 41 4	64 48 26	28 1 21	63 54 31	16 24 36		1·66302		3
27 3 31		1·68445		15 40 5	64 56 41	26 5 1	63 58 11	15 21 1		1·66095		4
25 15 21		1·68448		14 40 21	65 8 31	24 12 1	64 5 11	14 18 26		1·65883		5
23 30 46	3	1·68458	4	13 41 58	65 23 56	22 22 21	64 15 31	13 16 57	4	1·65667	3	6
21 49 11		1·68453		12 44 51	65 42 21	20 36 11	64 29 21	12 16 45		1·65459		7
20 11 11		1·68448		11 49 16	66 4 21	18 53 6	64 46 16	11 17 45		1·65240		8
18 36 46		1·68455		10 55 19	66 29 56	17 13 41	65 6 51	10 20 18		1·65037		9
17 5 26		1·68446		10 2 51	66 58 36	15 37 16	65 30 26	9 24 11		1·64818		10
15 37 50	4	1·68452	5	9 12 17	67 31 6	14 4 21	65 57 31	8 29 41	5	1·64603	4	11
14 13 41		1·68446		8 23 25	68 6 51	12 35 11	66 28 21	7 37 1		1·64400		12
12 53 26		1·68451		7 36 37	68 46 36	11 9 41	67 2 51	6 46 13		1·64203		13
11 36 41		1·68450		6 51 45	69 29 51	9 47 16	67 40 26	5 57 2		1·63984		14
10 23 56		1·68454		6 9 3	70 17 6	8 29 1	68 22 11	5 10 4		1·63782		15
9 14 51	4	1·68450	5	5 28 28	71 8 1	7 15 26	69 8 36	4 25 41	5	1·63615	4	16
8 9 51		1·68445		4 50 10	72 3 1	6 5 26	69 58 36	3 43 20		1·63433		17
7 9 6		1·68445		4 14 19	73 2 16	4 59 41	70 52 51	3 3 25		1·63258		18
6 12 51		1·68453		3 41 3	74 6 1	3 58 41	71 51 51	2 26 16		1 63100		19
5 20 21		1·68437		3 10 1	75 13 31	3 2 11	72 55 21	1 50 46		1·62956		20
4 33 1	5	1·68447	6	2 41 58	76 26 11	2 10 6	74 3 16	1 19 54	6	1·62807	5	21
3 49 36		1·68428		2 16 15	77 42 46	1 23 31	75 16 41	0 51 20		1·62691		22
3 12 1		1·68451		1 53 58	79 5 11	0 41 51	76 35 1	0 25 44		1·62583		23

TABLE III.

Outer sheet.						Inner sheet.						
ϕ.	$\delta\mu_2$.	μ_2.	$\delta\phi'$.	ϕ'.	$D+i$.	ϕ.	$D+i$.	ϕ'.	$\delta\phi'$.	μ_1.	$\delta\mu_1$.	
° ′ ″			″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″			
1 32 40		1·68464		0 55 0	73 39 50	0 49 20	71 17 50	0 30 20		1·62619		1
2 19 15		1·68450		1 22 39	72 26 25	0 4 30	70 2 40	0 2 46		1·62500		2
3 10 50		1·68454		1 47 28	71 18 0	0 44 25	68 51 35	0 27 21		1·62360		3
4 6 30		1·68450		2 26 15	70 13 40	1 37 25	67 44 35	1 0 3		1·62208		4
5 6 45		1·68458		3 1 56	69 13 55	2 34 45	66 41 55	1 35 28		1·62058		5
6 10 50	9	1·68456	9	3 30 52	68 18 0	3 35 50	65 43 0	2 13 26	9	1·61904	9	6
7 19 10		1·68464		4 20 14	67 26 20	4 40 40	64 47 50	2 53 26		1·61715		7
8 31 10		1·68462		5 2 41	66 38 20	5 49 5	63 56 15	3 35 53		1·61525		8
9 46 40		1·68449		5 47 9	65 53 50	7 0 50	63 8 0	4 20 28		1·61320		9
11 6 20		1·68462		6 33 56	65 13 30	8 16 20	62 23 30	5 7 23		1·61120		10
12 29 5	8	1·68453	8	7 22 24	64 36 15	9 34 45	61 41 55	5 56 12	8	1·60897	8	11
13 55 45		1·68462		8 12 55	64 2 55	10 56 30	61 3 40	6 47 4		1·60671		12
15 25 25		1·68457		9 5 2	63 32 35	12 21 35	60 28 45	7 39 59		1·60450		13
16 58 30		1·68454		9 58 50	63 5 40	13 49 10	59 56 20	8 34 30		1·60201		14
18 35 5		1·68462		10 54 19	62 42 15	15 19 40	59 26 50	9 30 46		1·59943		15
20 14 50	8	1·68466	7	11 51 14	62 22 0	16 53 35	59 0 45	10 29 0	7	1·59710	8	16
21 57 25		1·68459		12 49 27	62 4 35	18 29 20	58 36 30	11 28 25		1·59427		17
23 43 35		1·68465		13 49 6	61 50 45	20 8 30	58 15 40	12 29 39		1·59173		18
25 32 55		1·68469		14 49 58	61 40 5	21 50 30	57 57 40	13 32 24		1·58911		19
27 25 30		1·68475		15 51 58	61 32 40	23 35 15	57 42 25	14 36 32		1·58651		20
29 21 0	7	1·68457	6	16 54 56	61 28 10	25 22 10	57 29 20	15 41 47	6	1·58365	7	21
31 20 30		1·68464		17 59 3	61 27 40	27 12 0	57 19 10	16 48 24		1·58086		22
33 23 30		1·68465		19 4 4	61 30 40	29 4 50	57 12 0	17 56 15		1·57815		23
35 30 0		1·68458		20 9 52	61 37 10	31 0 20	57 7 30	19 5 10		1·57536		24
37 40 55		1·68465		21 16 33	61 48 5	32 59 5	57 6 15	20 15 13		1·57269		25
39 56 20		1·68473		22 23 57	62 3 10	35 0 50	57 8 0	21 26 12		1·56994		26

TABLE IV.

Outer sheet.					ϕ.	Inner sheet.					
$\delta\mu_2$.	μ_2.	$\delta\phi'$.	ϕ'.	$D+i$.		$D+i$.	ϕ'.	$\delta\phi'$.	μ_1.	$\delta\mu_1$.	
		″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″			
7	1·68444	6	18 16 33	61 27 40	31 53 10	57 6 50	19 36 20	6	1·57426	7	1
	1·68456		19 19 40	61 31 40	33 53 10	57 6 35	20 46 52		1·57144		2
	1·68464		20 21 44	61 38 55	35 53 10	57 9 40	21 56 27		1·56879		3
	1·68454		21 22 47	61 49 5	37 53 10	57 15 45	23 5 3		1·56616		4
	1·68454		22 22 32	62 2 40	39 53 10	57 25 0	24 12 31		1·56383		5
6	1·68448	7	23 21 2	62 19 20	41 53 10	57 37 15	25 18 47	7	1·56153	6	6
	1·68458		24 18 0	62 39 35	43 53 10	57 52 25	26 23 47		1·55929		7
	1·68452		25 13 38	63 2 45	45 53 10	58 10 40	27 27 22		1·55716		8
	1·68440		26 7 44	63 29 0	47 53 10	58 32 0	28 29 26		1·55512		9
	1·68452		27 0 1	63 59 0	49 53 10	58 57 5	29 29 33		1·55342		10
8	1·68452	8	27 50 39	64 32 10	51 53 10	59 24 50	30 28 13	8	1·55157	8	11
	1·68451		28 39 27	65 8 45	53 53 10	59 56 20	31 24 47		1·54996		12
	1·68454		29 26 17	65 49 0	55 53 10	60 31 20	32 19 19		1·54846		13
	1·68450		30 11 9	66 32 45	57 53 10	61 10 0	33 11 39		1·54708		14
	1·68459		30 53 49	67 20 25	59 53 10	61 52 0	34 1 58		1·54561		15
8	1·68448	9	31 34 29	68 11 35	61 53 10	62 38 45	34 49 19	9	1·54460	8	16
	1·68454		32 12 39	69 7 5	63 53 10	63 29 5	35 34 26		1·54348		17
	1·68464		32 48 23	70 6 50	65 53 10	64 23 50	36 16 45		1·54251		18
	1·68461		33 21 47	71 10 25	67 53 10	65 22 55	36 56 17		1·54163		19
	1·68452		33 52 43	72 18 15	69 53 10	66 26 35	37 32 52		1·54082		20
9	1·68465	10	34 20 43	73 31 5	71 53 10	67 35 10	38 6 15	10	1·54019	9	21
	1·68442		34 46 29	74 47 40	73 53 10	68 48 15	38 36 45		1·53948		22
	1·68447		35 9 5	76 9 35	75 53 10	70 6 45	39 3 42		1·53900		23
	1·68452		35 28 49	77 36 20	77 53 10	71 30 5	39 27 32		1·53846		24
	1·68446		35 45 48	79 7 45	79 53 10	72 59 10	39 47 36		1·53821		25
9	1·68444	11	35 59 45	80 44 15	81 53 10	74 33 10	40 4 31	11	1·53774	9	26

Section II.—*Determination of the Position of the Planes of the Prism* P R′ *with reference to the Axes of Elasticity.*

In order to compare these results with theory we require to know accurately the position of the principal planes of the two prisms. We will consider the prism, faces P R, first.

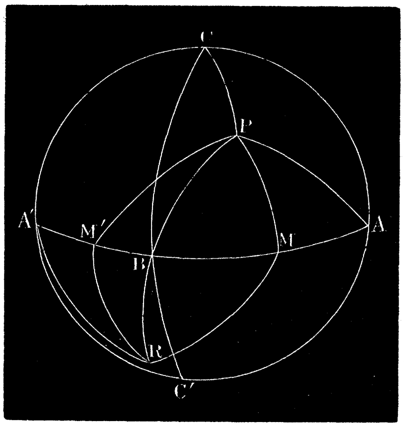
Let O A, O B, O C be the directions of the principal axes of the wave surface, O B bisects the angle between the faces m m', O C is parallel to the intersection of these faces.

Let the normals to the faces m m' cut a unit sphere, centre O, in M M′, the normal to the face P in P, and the normal to R in R.

Let the direction angles of P and R with reference to A, B, C be α_1 β_1 γ_1, α_2 β_2 γ_2 respectively.

[P lies in quadrant A B C, R in A′ B C′.]

Fig. 3.



Let

$$PM=\theta_1,\ PM'=\theta_1'$$
$$RM=\theta_2,\ RM'=\theta_2'$$

Observations on the angles between P and m, P and m' give θ_1, θ_1' respectively.

While observations on the angles between R and m, R and m' give θ_2, θ_2' respectively.

Let

$$MM'=2\mu=2MB.$$

Then, as in the first part, from triangles M′ B P, M P B we get

$$\cos\beta_1=\frac{\cos\frac{\theta_1'+\theta_1}{2}\cos\frac{\theta_1'-\theta_1}{2}}{\cos\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\cos\alpha_1=\frac{\sin\frac{\theta_1'+\theta_1}{2}\sin\frac{\theta_1'-\theta_1}{2}}{\cos\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

with similar formulæ for β_2, α_2.

All the angles θ_1, θ_1', θ_2, θ_2', and μ were observed several times throughout the work.

The mean of the observations give

$$\theta_1 = 54° \ 17' \ 6'' \quad \quad \quad (3)$$

$$\theta_1' = 75° \ 16' \ 20'' \quad \quad \quad (4)$$

$$\theta_2 = 88° \ 40' \ 35'' \quad \quad \quad (5)$$

$$\theta_2' = 88° \ 20' \ 20'' \quad \quad \quad (6)$$

$$\mu = 31° \ 53' \ 30'' \quad \quad \quad (7)$$

In no case was the difference between the mean and any one of the observations so great as 20″.

Substituting these values we find

$$\left.\begin{array}{l} \alpha_1 = 71° \ 49' \ 30'' \\ \beta_1 = 60° \ 25' \ 49'' \end{array}\right\} \quad \quad \quad (10)$$

$$\left.\begin{array}{l} \alpha_2 = 90° \ 19' \ 10'' \\ \beta_2 = 88° \ 14' \ 32'' \end{array}\right\} \quad \quad \quad (11)$$

Also

$$\sin^2 \gamma_1 = \cos^2 \alpha_1 + \cos^2 \beta_1$$
$$\sin^2 \gamma_2 = \cos^2 \alpha_2 + \cos^2 \beta_2$$

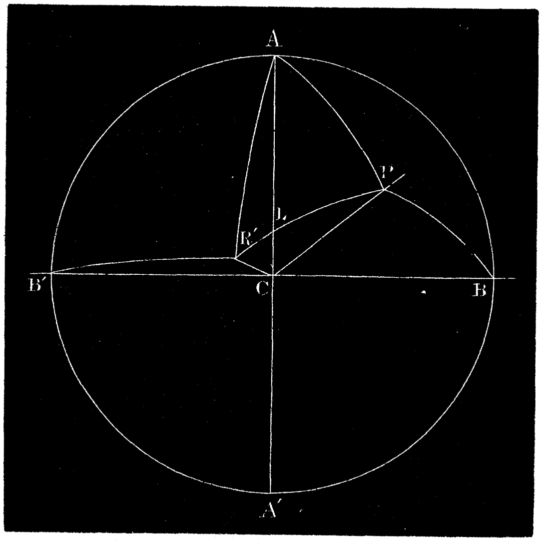
whence

$$\gamma_1 = 35° \ 42' \ 57'' \quad \quad \quad (12)$$

$$\pi - \gamma_2 = 1° \ 47' \ 11'' \quad \quad \quad (13)$$

Let R O meet the sphere again in R′, and let the figure represent the sphere as seen by an eye on O C produced.

Fig. 4.



We have now sufficient data to calculate the angle R′ P.

$$\begin{aligned} CP &= \gamma_1 \\ AP &= \alpha_1 \\ BP &= \beta_1 \\ CR' &= \pi-\gamma_2 \\ AR' &= \pi-\alpha_2 = 89^\circ\ 40'\ 50'' \\ B'R' &= \beta_2. \end{aligned}$$

From triangle B′ A R′

$$\cos B'R' = \cos B'AR' \sin AR'$$

$$\therefore \cos B'AR' = \frac{\cos B'R'}{\sin AR'}$$

$$= \frac{\cos \beta_2}{\sin \alpha_2}$$

whence

$$B'AR' = 88^\circ\ 14'\ 32''$$

$$\therefore R'AC = 1^\circ\ 45'\ 28''$$

From triangle C A P

$$\cos CP = \cos CAP \sin AP$$

$$\therefore \cos CAP = \frac{\cos CP}{\sin AP}$$

$$\therefore CAP = 31^\circ\ 17'\ 20''$$

$$\therefore R'AP = 33^\circ\ 2'\ 48''$$

From triangle R′ A P

$$\cos R'P = \cos R'A \cos AP + \sin R'A \sin AP \cos R'AP$$

$$= \cos(\pi-\alpha_2)\cos\alpha_1 + \sin(\pi-\alpha_2)\sin\alpha_1 \cos R'AP$$

whence

$$R'P = 37^\circ\ 2'\ 53''$$

But R′P can be measured directly, the mean of twelve observations, none of which differed from the mean by 10″, gave

$$R'P = 37^\circ\ 2'\ 56''$$

The close agreement between these results confirms the accuracy of the values of α_1, β_1, &c.

Let P R′ cut C A in L, we proceed to determine C L, R′ L, and the angle A L P.

From triangle A R′ L

$$\cot AL \sin AR' = \cot AR'P \sin R'AC + \cos AR' \cos R'AC$$

$$\therefore \cot AL \sin \alpha_2 = \cot AR'P \sin R'AC + \cos \alpha_2 \cos R'AC$$

we have seen already that

$$R'AC=1^\circ\ 45'\ 28''$$

For A R′ P we have

$$\sin AR'P=\frac{\sin AP}{\sin R'P}\sin R'AP$$

$$\therefore AR'P=59^\circ\ 18'\ 37''$$

On substituting we get

$$AL=88^\circ\ 38'\ 18''$$

$$\therefore CL=1^\circ\ 21'\ 42'' \quad \ldots\ldots\ldots\ldots \quad (15)$$

For R′ L the triangle R′ A L gives

$$\sin R'L=\frac{\sin R'AL \sin AL}{\sin AR'L}$$

whence

$$R'L=2^\circ\ 2'\ 37''$$

For A L P the triangle A L P gives

$$\sin ALP=\frac{\sin AP}{\sin LP}\sin LAP$$

$$LP=R'P-R'L=35^\circ\ 0'\ 19'' \quad \ldots\ldots\ldots\ldots \quad (16)$$

Whence

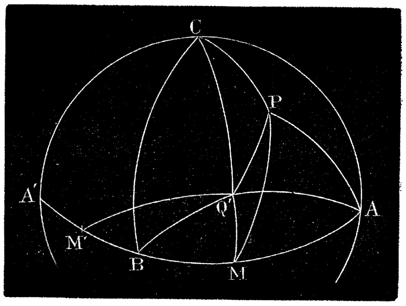
$$ALP=59^\circ\ 20'\ 11'' \quad \ldots\ldots\ldots\ldots \quad (17)$$

Thus we have determined the position of the plane R′ P completely.

Section III.—*Determination of the Position of the Plane* P Q *with reference to the Principal Planes.*

Our next step will be to determine the position of P Q′, Q′ being the point in which Q O produced backwards cuts the sphere.

Fig. 5.



Let the direction angles of O Q′ be $\alpha_3\ \beta_3\ \gamma_3$

Let

$$MQ'=\theta_3$$
$$M'Q'=\theta_3'$$

then

$$\cos\beta_3=\frac{\cos\frac{\theta_3'+\theta_3}{2}\cos\frac{\theta_3'-\theta_3}{2}}{\cos\mu}$$

$$\cos\alpha_3=\frac{\sin\frac{\theta_3'+\theta_3}{2}\sin\frac{\theta_3'-\theta_3}{2}}{\cos\mu}$$

But θ_3, θ_3' are the angles between the faces Q and m, Q and m' respectively. And observation shows that

$$\left.\begin{array}{l}\theta_3=19^\circ\ \ 0'\ 15''\\ \theta_3'\ \ 64^\circ\ 37'\ \ 0''\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Whence

$$\beta_3=35^\circ\ 58'\ 41'' \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\alpha_3=60^\circ\ 42'\ 56'' \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\cos Q'BA=\frac{\cos\alpha_3}{\sin\beta_3}$$

$$Q'BA=33^\circ\ 37'\ 51''$$

$$\cos CQ'=\sin BQ'\sin Q'BA$$

$$\therefore\gamma_3=CQ'=71^\circ\ 0'\ 44'' \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

To corroborate these values let us find P Q′ and compare with experiment

$$\cos Q'AB=\frac{\cos\beta_3}{\sin\alpha_3}$$

$$\therefore Q'AB=21^\circ\ 54'\ 12''$$

But we know from the previous work (Section II.) that

$$PAB=90^\circ-PAC=58^\circ\ 42'\ 42''$$

$$\therefore PAQ'=36^\circ\ 48'\ 30''$$

$$\cos PQ'=\cos AP\cos AQ'+\sin AP\sin AQ'\cos PAQ$$

Whence

$$PQ'=35^\circ\ 18'\ 35''$$

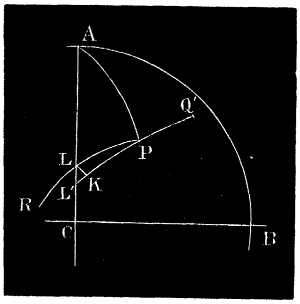
The mean of a large series of experiments gave

$$PQ'=35^\circ\ 18'\ 50''$$

and again we have strong evidence in favour of the correctness of the work.

Let the figure, as before, represent the sphere as seen by an eye on O C produced.

Fig. 6.



Let Q′ P produced cut C A in L′ we proceed to determine C L′, P L′, and P L′ A.

From the triangle L′ A P

$$\cot AL' \sin AP = \cot APL' \sin L'AP + \cos AP' \cos L'AP$$

From triangle P Q′ A.

$$\sin APL' = \frac{\sin AQ'}{\sin PQ'} \sin PAQ'$$

$$\pi - APL' = 64° \; 41' \; 28''$$

Also

$$L'AP = CAP = 31° \; 17' \; 20''$$

[Section II. from previous work.]

Substituting these values we get

$$AL' = 88° \; 44' \; 20''$$

$$CL' = 1° \; 15' \; 40'' \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

The triangle A L′ P gives

$$\sin AL'P = \frac{\sin AP}{\sin AL'} \sin APL'$$

whence

$$AL'P = 59° \; 13' \; 2'' \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

Also

$$\sin L'P = \frac{\sin AL'}{\sin APL'} \sin L'AP$$

whence

$$L'P = 35° \; 3' \; 14'' \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

To corroborate these results draw L K perpendicular to P L′, then L K L′ is approximately a right-angled triangle, and angle $LL'K = 60°$ nearly.

Hence

$$L'K = \tfrac{1}{2} LL' \text{ approximately}$$

$$\text{Now } L'K = PL' - PL = 3' \text{ approximately}$$

$$LL' = CL - CL' = 6' \text{ approximately}$$

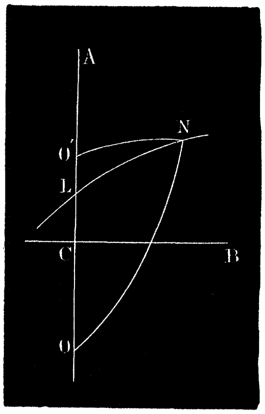
$$\therefore L'K \text{ is equal to } \tfrac{1}{2} LL'$$

Section IV.—*General Theory.*

To apply these results to the theoretical calculation.

Taking the same figure

Fig. 7.



Let the optic axes cut the sphere in O O′; consider a plane cutting the plane A O C between C and O′ in L say; let N be any point on it; let NLA$=\chi$, NO$=\theta$, NO′$=\theta'$.

Then we know that if v_1, v_2 are the velocities along the wave normal through N, a, b, c the principal velocities, a being the greatest

$$v_1^2-v_2^2=(a^2-c^2)\sin\theta\sin\theta'$$

$$v_1^2+v_2^2=(a^2+c^2)-(a^2-c^2)\cos\theta\cos\theta'$$

$$2v_1^2=(a^2+c^2)-(a^2-c^2)\cos(\theta+\theta') \quad \ldots \quad (1)$$

$$2v_2^2=(a^2+c^2)-(a^2-c^2)\cos(\theta-\theta'). \quad \ldots \quad (2)$$

We require, therefore, the values of θ and θ'.

Now

$$\cos\theta=\cos\mathrm{OL}\cos\mathrm{LN}-\sin\mathrm{OL}\sin\mathrm{LN}\cos\chi$$

$$=\cos\mathrm{OL}(\cos\mathrm{LN}-\tan\mathrm{OL}\sin\mathrm{LN}\cos\chi)$$

$$=\frac{\cos\mathrm{OL}}{\cos\lambda}\cos(\mathrm{LN}+\lambda) \quad \ldots \quad (3)$$

if $\tan\lambda=\tan\mathrm{OL}\cos\chi$

Similarly

$$\cos\theta'=\frac{\cos\mathrm{O'L}}{\cos\lambda'}\cos(\mathrm{LN}-\lambda') \quad \ldots \quad (4)$$

where

$$\tan\lambda'=\tan\mathrm{O'L}\cos\chi.$$

Thus to use the formulæ for v_1, v_2 we require to know accurately the position of the optic axes.

These can only be determined by finding the values of a, b, c.

The results of the experiments enable us to do this with considerable exactness.

Let μ_a, μ_b, μ_c be the principal refractive indices, μ_a being the greatest, μ_c the least

$$\mu_a = \frac{1}{c}$$

$$\mu_b = \frac{1}{b}$$

$$\mu_c = \frac{1}{a}$$

Section V.— *Values of the Principal Refractive Indices.*

We are now in a position to determine from our experimental results the values of a, b, c the principal velocities.

The plane P R′ cuts the plane A C at a short distance from C.

The section of the surface of wave slowness by the plane A O C consists of a circle of radius μ_b and an oval curve of axes μ_a and μ_c in the directions O C, O A respectively, which for a small distance on either side of a principal axis may be treated as an ellipse; as in FRESNEL's theory.

Also, if L be the point in which P R′ cuts the plane A O C

$$LP = 35^\circ\ 0'\ 19''$$

[*vide* Section II. (16)].

The value of μ corresponding to a value of $\phi' = 35^\circ\ 0'\ 19''$ in Table I. for the inner sheet will thus be the value of μ_b.

But we have as corresponding values

ϕ'	μ
34° 28′ 43″	1·68115
34° 53′ 57″	1·68114
35° 16′ 20″	1·68115

we may therefore take

$$\mu_b = 1{\cdot}68115.$$

The values of the radius vector to the outer sheet in the same neighbourhood are, Table I., lines 4 and 5,

ϕ'	μ
34° 47′ 54″	1·68546
35° 10′ 1″	1·68553.

For

$$\phi' = 35^\circ\ 0'\ 19''$$

we may take therefore

$$\mu=1\cdot68550$$

This then must be the length of the radius vector of an ellipse, axes μ_a and μ_c, which passes through the point L—*i.e.*, which is inclined at an angle CL=1° 21′ 42″ to the axis μ_a [*vide* Section II. (15)].

Now if θ be the angle which a radius vector r makes with the axis μ_a

$$\frac{1}{r^2}=\frac{\cos^2\theta}{\mu_a^2}+\frac{\sin^2\theta}{\mu_c^2}$$

in the case considered

$$\theta=1^\circ\ 21'\ 42''$$

$$\sin^2\theta=\cdot000565$$

and the error made in giving to μ_c an approximate value will be inappreciable.

Let us take, therefore, for μ_c the value found in the previous part which agrees with RUDBERG's value

$$\mu_c=1\cdot53013$$

$$\frac{\sin^2\theta}{\mu_c^2}=\cdot000241$$

Now

$$r=1\cdot68550$$

$$\frac{1}{r^2}=\cdot351919$$

$$\frac{1}{\mu_a^2}=\frac{1}{\cos^2\theta}\left(\frac{1}{r^2}-\frac{\sin^2\theta}{c^2}\right)$$

whence

$$\mu_a=\underline{1\cdot68560}$$

It now remains to determine μ_c accurately.

The natural faces $a_{,}m_{,}'$ (fig. 1) formed a prism whose edge was parallel to the axis O C.

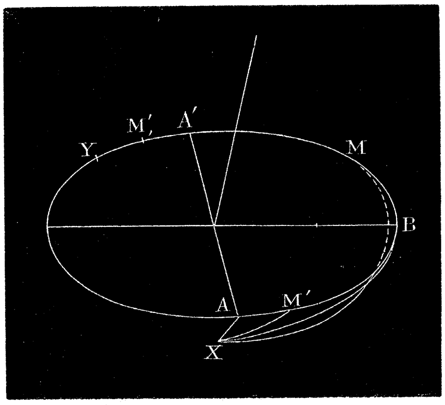
By passing light through these faces and observing the incidence and deviation as before, I was enabled to calculate μ_c.

To pass light through these faces it was necessary, however, to polish them, and this operation altered their position appreciably. The values obtained by direct measurement required a small correction, and to calculate this it was necessary to determine the position of these new faces.

Let the faces approximately coincident with $a_{,}$ and $m_{,}'$ be called X Y respectively. On levelling the faces of reference $m\ m'$, I found that Y remained very nearly in the same zone as before, viz.: that of $m\ m'\ m_{,}\ m_{,}'$; while X fell rather below this zone. Observations on the angles between X and Y and the faces $m\ m'$ respectively gave

$$XM\ (=\theta_4 \text{ say}) = 123^\circ\ 32'$$
$$XM'(=\theta'_4 \text{ say}) = 59^\circ\ 52'$$
$$\text{arc } YA'M\ (=\theta_5 \text{ say}) = 117^\circ\ 17'$$
$$\text{arc } YA'M'(=\theta_5 \text{ say}) = 181^\circ\ 3'$$

Fig. 8.



Since Y, M, M', are in one zone with A and B, and B bisects arc M M'

$$2BA'Y = M'A'Y + MA'Y$$
$$\therefore \text{ arc } BA'Y = 149^\circ\ 10'$$
$$A'Y = 149^\circ\ 10' - 90^\circ$$
$$= 59^\circ\ 10'$$

Let $\alpha_4\ \beta_4\ \gamma_4$ be the direction angles of X.
Then, as before, [Section II. (1)]

$$\cos\alpha_4 = \frac{\sin\frac{\theta'_4+\theta_4}{2}\sin\frac{\theta'_4-\theta_4}{2}}{\cos\mu} \quad \&c.$$

whence

$$\alpha_4 = 3^\circ\ 28'\ 50''$$
$$\beta_4 = 91^\circ\ 42'$$

To test these values let us calculate the angle X Y; the triangle X A B gives

$$\cos XAB = \frac{\cos XB}{\sin AX}$$

whence

$$\pi - XAB = 60^\circ\ 43'\ 50''$$

In the triangle X A Y we now know A X, A Y and the angle X A Y.

Solving, we get

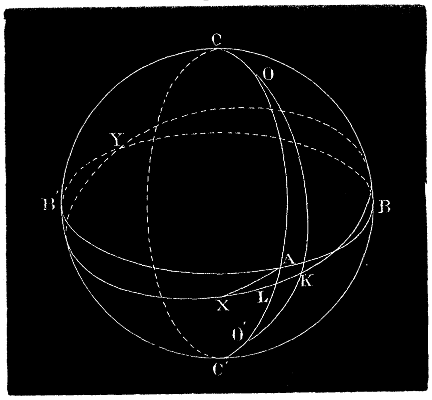
$$XY = 119° \ 5' \quad \ldots \ldots \ldots \quad (1)$$

Experiment gave

$$119° \ 4'$$

Here again we have considerable confirmation.

Fig. 9.



Let the plane Y X L be the plane of the prism cutting C A C′ in L; it will be necessary to find A L, X L, and the angle A L Y.

The right angled triangle L A Y gives

$$\sin AY = \tan AL \cot AYL.$$

The triangle X A Y gives

$$\sin LYA = \frac{\sin AX \sin YAX}{\sin XY}$$

whence

$$LYA = 3° \ 28' \ 30''$$

Substituting this value, we get

$$AL = 2° \ 59' \ 10'' \quad \ldots \ldots \ldots \quad (2)$$

From the same triangle we get

$$\cos YL = \cos AY \cos AL$$

whence

$$YL = 120° \ 47' \ 10''$$

but

$$XY = 119° \ 4'$$

$$\therefore XL = 1° \ 43' \ 10'' \quad \ldots \ldots \ldots \quad (3)$$

From the same triangle again

$$\cos ALY = \tan AL \cot LY$$

whence

$$ALY = 91° \ 47''$$

Now we know that if O, O′ are the points where the optic axes cut the unit sphere K, the point where any wave normal meets the sphere if

$$KO = \theta$$

$$KO' = \theta'$$

$$2v^2 = \frac{1}{\mu_a^2} + \frac{1}{\mu_c^2} - \left(\frac{1}{\mu_a^2} - \frac{1}{\mu_c^2}\right) \cos(\theta - \theta')$$

$$\therefore \frac{1}{\mu_c^2}(1 + \cos(\theta - \theta'))$$

$$= \frac{2}{\mu^2} - \frac{1}{\mu_a^2}(1 - \cos(\theta - \theta'))$$

$$\frac{1}{\mu_c^2} \cos^2\left(\frac{\theta - \theta'}{2}\right) = \frac{1}{\mu^2} - \frac{1}{\mu_a^2} \sin^2\left(\frac{\theta - \theta'}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If then we can determine $\theta - \theta'$ since μ is given by experiment, this formula determines the value of μ_c.

Now along the plane X Y, $\theta - \theta'$ is very small, and a considerable change in the positions of O O′ will produce but small variations in the value of $\theta - \theta'$ and $\therefore$ of μ_c.

Let us, therefore, take the value of the angle between the optic axes given by the values already determined for μ_a and μ_b, viz. :

$$\mu_a = 1{\cdot}68560$$

$$\mu_b = 1{\cdot}68115$$

and the approximate value of μ_c

$$\mu_c = 1{\cdot}53013$$

If $2\phi'$ is the angle between the optic axes

$$\tan \phi' = \frac{\mu_c}{\mu_a} \sqrt{\left\{\frac{\mu_a^2 - \mu_b^2}{\mu_b^2 - \mu_c^2}\right\}}$$

whence

$$\phi' = 9^\circ \ 4' \ 5''$$

Hence

$$AO = AO' = 80^\circ \ 55' \ 55''$$

$$AL = \ 2^\circ \ 59' \ 10''$$

$$OL = 83^\circ \ 55' \ 5''$$

$$O'L = 77^\circ \ 56' \ 45''$$

$$\chi = ALK = 88^\circ \ 13'$$

$$XL = 1^\circ \ 43' \ 10''$$

Let K be on the side of L remote from X. Experiment shows that this is the actual position of the wave normal in the case considered.

Let

$$KX=\psi'$$

$$\therefore LK=KX-XL=\psi'-1^\circ\ 43'\ 10''$$

$$\cos\theta=\cos OL\cos LK+\sin OL\sin LK\cos\chi=\frac{\cos OL}{\cos\lambda}\cos(LK-\lambda)$$

if

$$\tan\lambda=\tan OL\cos\chi$$

Substituting the values of O L and χ

$$\lambda=16^\circ\ 17'\ 10''$$

$$\log\frac{\cos OL}{\cos\lambda}=\bar{1}\cdot 0427912$$

$$LK-\lambda=\psi'-18^\circ\ 0'\ 20''$$

Similarly

$$\cos\theta'=\frac{\cos O'L}{\cos\lambda}\cos(LK+\lambda')$$

where

$$\tan\lambda'=\tan O'L\cos\chi$$

$$\therefore \lambda'=8^\circ\ 17'\ 30''$$

$$\log\frac{\cos O'L}{\cos\lambda'}=\bar{1}\cdot 3243204$$

$$LK+\lambda'=\psi'+6^\circ\ 34'\ 20''$$

A table on the next page gives the values of ψ, the angle made by the incident wave normal with the normal to X; $D+i$ the deviation + the angle X Y; ψ' the angle made by the wave normal in the crystal with the normal to X; and $\frac{1}{\mu^2}$, μ being the reciprocal of the wave velocity.

TABLE to determine the Value of μ_c.

ψ.	D + i.	ψ'.	$\frac{1}{\mu^2}$.
51° 16′ 7″	101° 47′ 15″	30° 39′ 0″	·427076
49 41 7	101 48 45	29 53 13	·427061
47 49 7	101 57 45	28 57 43	·427049
46 5 22	102 13 0	28 6 3	·427039
44 26 7	102 34 45	27 14 5	·427029
42 55 27	103 3 5	26 25 31	·427018

Substituting the values of ψ' from this table in the formulæ for cos θ and cos θ' we can find θ and θ'.

The values of $\theta-\theta'$ are

3° 31′ 50″
3 37 30
3 42 20
3 47 50
3 52 50
3 57 40

Substituting these values in the formula for μ_c, the mean of the results agrees closely with

$$\mu_c=1·53013$$

Thus we may take

$$\mu_a=1·68560$$

$$\mu_b=1·68115$$

$$\mu_c=1·53013$$

And the angle between O C and the optic axes is

$$9° \; 4' \; 5'' \quad \ldots \ldots \quad (5)$$

If ϕ is the angle as seen in air

$$\phi=15° \; 21' \; 50''$$

The value KIRCHHOFF found by experiment was

$$15° \; 27'$$

The close agreement is noteworthy.

As in the first part, the agreement is much closer than it would be taking RUDBERG's values of μ_a, μ_b, μ_c.

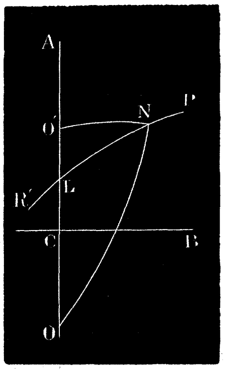
Section VI.—*Application of the Theory to Experiment.—Tables giving Theoretical Values of* μ.—*Discussion of Results.*

We can now continue the work of calculating the theoretical values of the reciprocal of the velocity.

Consider first the plane P R′.

With the same notation we have [Section IV. (3 and 4)].

Fig. 10.



$$\cos\theta = \frac{\cos OL}{\cos\lambda}\cos(NL+L) \quad \text{. . . . . (1)}$$

$$\tan\lambda = \tan OL \cos ALN$$

$$\cos\theta' = \frac{\cos O'L'}{\cos\lambda'}\cos(NL-\lambda') \quad \text{. . . . . (2)}$$

$$\tan\lambda' = \tan O'L \cos ALN$$

$$OC=O'C=9^\circ\ 4'\ 5'' \quad \text{[Section V. (5)]}$$

$$LC=1^\circ\ 21'\ 42'' \quad \text{[Section II. (15)]}$$

$$\therefore OL=10^\circ\ 25'\ 47'' \quad \text{. . . . . (3)}$$

and

$$O'L=7^\circ\ 42'\ 23'' \quad \text{. . . . . (4)}$$

$$ALN=\chi=59^\circ\ 20'\ 11'' \quad \text{[Section II. (17)].}$$

Whence

$$\lambda=5^\circ\ 21'\ 47'' \quad \text{. . . . . (5)}$$

$$\log\frac{\cos OL}{\cos\lambda}=\bar{1}\cdot 9946698 \quad \text{. . . . . (6)}$$

$$\lambda'=3^\circ\ 56'\ 52'' \quad \text{. . . . . (7)}$$

$$\log\frac{\cos O'L}{\cos\lambda'}=\bar{1}\cdot 9970915 \quad \text{. . . . . (8)}$$

$$LN = LP - PN = LP - \phi' = 35^\circ\ 0'\ 19'' - \phi' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$LN + \lambda = 40^\circ\ 22'\ 6'' - \phi' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$LN - \lambda' = 31^\circ\ 3'\ 27'' - \phi' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Substituting these values we can calculate θ and θ', and hence find the values of v_1 and v_2 or their reciprocals μ_1 and μ_2.

Tables V. and VI. give the results of this substitution: on the right hand for the inner sheet, on the left hand for the outer.

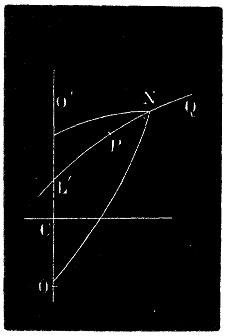
The centre columns give the values of L N+λ, L N−λ′.

Then on either side, $\theta+\theta'$ for the inner sheet, and $\theta-\theta'$ for the outer. Then the theoretical values of μ_1 μ_2, then the experimental values taken from Tables I. and II., and finally, in the outside column, the excess of theory above experiment.

The next step is to discuss the theory for the plane P Q.

With the same figure and notation as before, if this plane cut the plane A C in L′ we have [Section IV. (3 and 4)] $\lambda_{,}$ $\lambda'_{,}$ being values of λ λ' in this case.

Fig. 11.



$$\cos\theta = \frac{\cos OL'}{\cos\lambda_{,}} \cos(L'N + \lambda_{,}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$\tan\lambda_{,} = \tan OL' \cos AL'N$$

$$\cos\theta' = \frac{\cos O'L'}{\cos\lambda'_{,}} \cos(L'N - \lambda'_{,}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where

$$\tan\lambda'_{,} = \tan O'L' \cos AL'N$$

$$CO = CO' = 9^\circ\ 4'\ 5'' \quad \text{[Section V. (5)]}$$

$$CL' = 1^\circ\ 15'\ 40'' \quad \text{[Section III. (7)]}$$

$$\therefore OL' = 10^\circ\ 19'\ 45'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$O'L' = 7^\circ\ 48'\ 25'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$AL'N = \chi' = 59^\circ\ 13'\ 2'' \quad \text{[Section III. (8)]}$$

whence

$$\lambda_{,}=5^{\circ}\ 19'\ 45'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$\lambda_{,}'=4^{\circ}\ 0'\ 50''. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

L′N=L′P+PN (N being now on the side of P remote from L′) $=35^{\circ}\ 3'\ 14''+\phi'$.

$$\mathrm{L'N}+\lambda_{,}=40^{\circ}\ 22'\ 59''+\phi' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$\mathrm{L'N}-\lambda_{,}'=31^{\circ}\ 2'\ 24''+\phi' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$\log\frac{\cos \mathrm{OL'}}{\cos\lambda_{,}}=\bar{1}\cdot 9947864 \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$\log\frac{\cos \mathrm{O'L'}}{\cos\lambda_{,}'}=\bar{1}\cdot 9970226 \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and as before, we can calculate from these data the values of θ, θ' and hence of μ_1 μ_2.

Tables VII. and VIII give the results of this.

The fifth column gives the values of L N+λ for the inner sheet, the next L N−λ′, the next $\theta+\theta'$, the eighth the value of the reciprocal of the velocity calculated on FRESNEL's theory [by means of Section IV. (1 and 2)],

$$\frac{2}{\mu_1^{\,2}}=a^2+c^2-(a^2-c^2)\cos(\theta+\theta')$$

$$\frac{2}{\mu_2^{\,2}}=a^2+c^2-(a^2-c^2)\cos(\theta-\theta')$$

the ninth, the same quantity from experiment, and the last, the excess of the theoretical over the experimental value.

The values of $\theta-\theta'$ given in the fourth column are calculated for the inner sheet.

Now as the values of ϕ' corresponding to the same incident wave are different for the two sheets, the values of θ and θ' will be different; but on referring to the values it will be seen that after the first twenty observations, $\theta-\theta'$ is very nearly constant throughout the arc considered; so that μ_2 varies very slowly, and except just at first, we may, to the degree of approximation required, use the value of θ and θ' for the inner sheet in calculating the theoretical value of the reciprocal of the velocity for the outer sheet.

In the case of the first twenty observations for which $\theta-\theta'$ varies appreciably with ϕ', the experimental values of μ_2 are obtained by interpolation from those values in Table I., to which they correspond.

Table V. contains the theory for I.
,, VI. ,, ,, II.
,, VII. ,, ,, III.
,, VIII. ,, ,, IV.

TABLE V.

Outer sheet.				LN + λ.	LN − λ′.	Inner sheet.				
Difference.	μ_2 Experiment.	μ_2 Theory.	θ − θ′.			θ + θ′.	μ_1 Theory.	μ_1 Experiment.	Difference.	
			° ′ ″	° ′ ″	° ′ ″	° ′ ″				
				4 15 50	5 2 49	18 14 0	1·68111	1·68102	09	1
				4 29 36	4 49 3	18 11 50	1·68113	1·68112	01	2
− 4	1·68558	1·68554	2 6 50	4 46 7	4 32 32	18 9 50	1·68114	1·68106	08	3
− 2	1·68553	1·68551		5 5 46	4 12 53	18 8 30	1·68115	1·68115	00	4
+ 1	1·68546	1·68547		5 28 9	3 50 30	18 8 10	1·68115	1·68114	01	5
+ 5	1·68539	1·68544		5 53 23	3 25 16	18 9 40	1·68114	1·68115	−01	6
+ 9	1·68532	1·68541		6 21 12	2 57 27	18 13 10	1·68111	1·68106	05	7
+10	1·68527	1·68537	4 12 20	6 51 42	2 26 57	18 19 40	1·68106	1·68099	07	8
+ 9	1·68520	1·68529		7 24 45	1 53 54	18 29 50	1·68097	1·68089	08	9
+20	1·68502	1·68522		8 0 14	1 18 25	18 44 30	1·68085	1·68074	09	10
+ 0	1·68515	1·68515		8 38 15	0 40 24	19 4 30	1·68068	1·68069	−01	11
+ 9	1·68498	1·68507		9 18 10	0 0 29	19 30 50	1·68046	1·68030	16	12
+ 8	1·68490	1·68498	6 44 30	10 0 38	0 41 59	20 3 50	1·68017	1·68016	01	13
+ 3	1·68486	1·68489		10 44 46	1 26 7	20 44 10	1·67980	1·67962	18	14
+16	1·68465	1·68481		11 31 5	2 12 26	21 32 50	1·67936	1·67917	19	15
+ 1	1·68473	1·68474		12 20 13	3 1 34	22 29 0	1·67880	1·67859	21	16
+ 4	1·68463	1·68467		13 9 18	3 50 39	23 31 30	1·67817	1·67806	11	17
− 8	1·68454	1·68462	8 28 10	14 0 52	4 42 13	24 42 10	1·67743	1·67721	22	18
+ 1	1·68457	1·68458		14 54 8	5 35 29	26 0 30	1·67656	1·67634	22	19
− 3	1·68457	1·68454		15 48 55	6 30 16	27 23 50	1·67560	1·67532	28	20
− 8	1·68459	1·68451		16 45 11	7 26 32	28 53 10	1·67463	1·67421	42	21
− 3	1·68451	1·68448		17 42 53	8 24 14	30 28 20	1·67331	1·67303	27	22
− 3	1·68448	1·68445	9 10 40	18 41 48	9 23 9	32 7 0	1·67200	1·67161	39	23
	1·68441			19 42 3	10 23 24	33 52 0	1·67054	1·67015	39	24
	1·68446			20 43 28	11 24 49	35 40 30	1·66896	1·66854	42	25
	1·68461			21 46 6	12 27 27	37 33 20	1·66726	1·66694	32	26
	1·68446			22 50 40	13 31 1	39 26 50	1·66547	1·66502	45	27
− 9	1·68448	1·68439	9 26 20	23 54 16	14 35 37	41 26 0	1·66352	1·66300	52	28

TABLE VI.

Outer sheet.				LN + λ.	LN − λ′.	Inner sheet.				
Difference.	μ_2 Experiment.	μ_2 Theory.	$\theta - \theta'$.			$\theta + \theta'$.	μ_1 Theory.	μ_1 Experiment.	Difference.	
			° ′ ″	° ′ ″	° ′ ″	° ′ ″				
	1·68433			21 47 46	12 29 7	37 35 30	1·66702	1·66678	24	1
	1·68458			22 53 5	13 34 26	39 34 50	1·66534	1·66490	44	2
	1·68446			23 58 30	14 38 51	41 32 40	1·66341	1·66302	39	3
	1·68445			25 1 5	15 42 26	43 29 30	1·66143	1·66095	48	4
	1·68448			26 3 40	16 45 1	45 26 0	1·65939	1·65883	56	5
−21	1·68458	1·68437	9 29 20	27 5 9	17 46 30	47 21 20	1·65730	1·65667	63	6
	1·68453			28 5 21	18 46 42	49 14 40	1·65520	1·65459	61	7
	1·68448			29 4 21	19 45 42	51 7 20	1·65306	1·65240	66	8
	1·68445			30 1 48	20 43 9	52 54 40	1·65098	1·65037	61	9
	1·68446			30 57 55	21 39 16	54 42 40	1·64884	1·64818	66	10
−15	1·68452	1·68437	9 30 40	31 52 25	22 33 46	56 26 50	1·64674	1·64603	71	11
	1·68446			32 45 5	23 26 26	58 8 30	1·64466	1·64400	66	12
	1·68451			33 35 53	24 17 14	59 45 20	1·64264	1·64203	61	13
	1·68450			34 25 4	25 6 25	61 20 10	1·64055	1·63984	69	14
	1·68454			35 12 2	25 53 23	62 50 40	1·63872	1·63782	90	15
−13	1·68450	1·68437	9 30 40	35 56 25	26 37 46	64 16 40	1·63678	1·63615	63	16
	1·68445			36 38 46	27 20 7	65 38 40	1·63509	1·63433	76	17
	1·68445			37 18 41	28 0 2	66 55 50	1·63340	1·63258	82	18
	1·68453			37 55 50	28 37 11	68 7 50	1·63182	1·63100	82	19
	1·68437			38 31 20	29 12 41	69 16 30	1·63030	1·62956	74	20
−10	1·68447	1·68437	9 30 0	39 2 12	29 43 33	70 16 40	1·62897	1 62807	90	21
	1·68428			39 30 46	30 12 7	71 12 20	1·62776	1·62691	85	22
	1·68451			39 56 22	30 37 43	72 2 0	1·62662	1·62583	79	23

Table VII.

Outer sheet.				LN + λ.	LN − λ'.	Inner sheet.				
Difference.	μ_2 Experiment.	μ_2 Theory.	$\theta - \theta'$.			$\theta + \theta'$.	μ_1 Theory.	μ_1 Experiment.	Difference.	
			° ′ ″	° ′ ″	° ′ ″	° ′ ″				
−27	1·68464	1·68437	9 29 50	39 52 39	30 32 4	71 52 50	1·62682	1·62619	63	1
	1·68450			40 20 13	30 59 38	72 46 30	1·62563	1·62500	63	2
	1·68454			40 50 20	31 29 45	73 45 10	1·62430	1·62360	70	3
	1·68450			41 23 2	32 2 27	74 48 50	1·62268	1·62208	60	4
	1·68458			41 58 27	32 37 52	75 58 0	1·62131	1·62058	74	5
−19	1·68456	1·68437	9 29 30	42 36 25	33 15 50	77 12 10	1·61965	1·61904	61	6
	1·68464			43 16 25	33 55 50	78 30 20	1·61786	1·61715	71	7
	1·68462			43 58 52	34 38 17	79 53 30	1·61600	1·61525	75	8
	1·68469			44 43 27	35 22 52	81 20 20	1·61400	1·61320	80	9
	1·68462			45 30 22	36 9 47	82 52 0	1·61192	1·61120	72	10
−15	1·68453	1·68438	9 28 30	46 19 11	36 58 36	84 27 50	1·60974	1·60897	77	11
	1·68462			47 10 3	37 49 28	86 7 40	1·60748	1·60671	77	12
	1·68457			48 2 58	38 42 23	87 51 10	1·60514	1·60450	64	13
	1·68454			48 57 29	39 36 54	89 38 10	1·60285	1·60201	84	14
	1·68462			49 53 45	40 33 10	90° + 1 28 40	1·60025	1·59943	82	15
−28	1·68466	1·68438	9 27 20	50 51 59	41 31 24	90° + 3 23 0	1·59769	1·59710	59	16
	1·68549			51 51 24	42 30 49	,, 5 19 50	1·59510	1·59427	83	17
	1·68465			52 52 38	43 32 3	,, 7 20 20	1·59246	1·59173	73	18
	1·68469			54 55 23	44 34 48	,, 9 23 40	1·58977	1·58911	66	19
	1·68475			54 59 31	45 38 56	,, 11 30 0	1·58705	1·58651	54	20
−18	1·68457	1·68439	9 26 20	56 4 46	46 44 11	90° +13 38 30	1·58432	1·58365	67	21
	1·68464			57 11 23	47 50 48	,, 15 49 30	1·58155	1·58086	69	22
	1·68465			58 19 14	48 58 39	,, 18 3 30	1·57880	1·57815	65	23
	1·68458			59 28 9	50 7 34	,, 20 19 30	1·57605	1·57536	69	24
	1·68465			60 38 12	51 17 37	,, 22 37 40	1·57329	1·57269	60	25
−36	1·68475	1·68439	9 24 30	61 49 11	52 28 36	90° +24 57 50	1·57058	1·56994	64	26

TABLE VIII.

Outer sheet.				LN + λ.	LN − λ′.	Inner sheet.				
Difference.	μ_2 Experiment.	μ_2 Theory.	θ − θ′.			θ + θ′.	μ_1 Theory.	μ_1 Experiment.	Difference.	
			° ′ ″	° ′ ″	° ′ ″	° ° ′ ″				
− 5	1·68444	1·68439	9 25 10	59 59 19	50 38 44	90+21 20 50	1·57482	1·57426	56	1
	1·68456			61 9 51	51 49 16	„ 23 40 0	1·57209	1·57144	65	2
	1·68464			62 19 26	52 58 51	„ 25 57 30	1·56944	1·56879	65	3
	1·68454			63 28 2	54 7 27	„ 28 13 0	1·56690	1·56616	74	4
	1·68454			64 35 30	55 14 55	„ 30 26 20	1·56446	1·56383	63	5
− 8	1·68448	1·68440	9 22 30	65 41 46	56 21 11	90+32 36 10	1·56215	1·56153	62	6
	1·68458			66 46 46	57 26 11	„ 34 45 50	1·55991	1·55929	62	7
	1·68452			67 50 21	58 29 46	„ 36 51 10	1·55779	1·55716	63	8
	1·68440			68 52 25	59 31 50	„ 38 54 20	1·55580	1·55512	68	9
	1·68452			69 52 32	60 31 57	„ 40 52 50	1·55393	1·55342	51	10
−12	1·68452	1·68440	9 21 50	70 51 12	61 30 47	90+42 48 50	1·55216	1·55157	59	11
	1·68451			71 47 46	62 27 11	„ 44 40 50	1·55051	1·54996	55	12
	1·68454			72 42 18	63 21 43	„ 46 28 40	1·54899	1·54846	53	13
	1·68450			73 34 38	64 14 3	„ 48 12 0	1·54756	1·54708	48	14
	1·68459			74 24 57	65 4 22	„ 49 51 40	1·54623	1·54561	62	15
− 7	1·68448	1·68441	9 20 0	75 11 18	65 51 43	90+51 24 20	1·54515	1·54460	55	16
	1·68454			75 57 25	66 36 50	„ 52 54 40	1·54393	1·54348	45	17
	1·68464			76 39 44	67 19 9	„ 54 18 20	1·54293	1·54251	42	18
	1·68461			77 19 16	67 58 41	„ 55 36 40	1·54203	1·54163	40	19
	1·68452			77 55 51	68 35 16	„ 56 49 0	1·54123	1·54082	41	20
−23	1·68465	1·68442	9 19 50	78 29 14	69 8 39	90+57 55 10	1·54051	1·54019	32	21
	1·68442			78 59 44	69 39 9	„ 58 55 30	1·53988	1·53948	40	22
	1·68447			79 26 41	70 6 6	„ 59 49 0	1·53934	1·53900	34	23
	1·68452			79 50 31	70 29 56	„ 60 36 10	1·53887	1·53846	41	24
	1·68446			80 10 35	70 50 0	„ 61 16 0	1·53848	1·53821	27	25
− 2	1·68444	1·68442	9 19 20	80 27 39	71 6 55	90+61 49 20	1·53817	1·53774	33	26

Let us consider first the inner sheet.

In Table V. we see the differences are small to begin with, and increase fairly regularly throughout the whole of the table, the experimental value being almost always less than the theoretical.

Lines 4, 5, and 6 were those chosen to determine the value of μ_b which explains the coincidence of the two curves in that neighbourhood. The table covers an arc of the surface of nearly 20°, as is seen by referring to the values of ϕ' in Table I.

Throughout this arc the theoretical section lies outside the experimental, the difference between the two increasing as we proceed from the axis μ_b.

This continues throughout Table VI.

The first three lines of Table VI. just overlap the last three of Table V. The difference reaches a maximum value towards the end of Table VI. at a point for which $\phi'=$ 1° 19′ 54″ (*vide* Tables II., 21, and VI., 21), that is at about 34° away from the axis μ_b.

After this the difference begins to decrease at first slowly throughout Table VII. which refers to the other prism P Q, so that the arc of section has changed slightly between Tables VI. and VII., both arcs passing through the point P.

Table VII. covers an arc of 22° (Table III.).

The first two lines in Table VII. refer to a wave normal lying on the same side of P as L. In the rest of the table the wave normal is on the side of P remote from L.

This arc is continued throughout Table VIII.; the theoretical values of μ are still greater than the experimental, but the difference diminishes as we proceed along the arc.

The last three observations in Table VII. and the first three in Table VIII. overlap.

Table VIII. covers an arc of 20°.

Thus the observations extend over two arcs, each passing through the point P inclined to one another at that point at so small an angle as to be almost continuous, the one arc being 35° 40′, and the other 40° 40′ in length.

The results of theory and experiment agree at the extremity of the first arc, differ most widely in the neighbourhood of the point where the two arcs meet, and tend towards equality again throughout the second arc.

It is worth noticing that the experimental results at the end of Table VI., the end of the first arc, agree closely with those at the beginning of Table VII., the beginning of the second arc.

The experimental results for the inner sheet are therefore represented by a curve which coincides with FRESNEL'S curve at the extremity of one axis, and lies inside it throughout the next 75° of its length.

For the outer sheet the differences are less than for the inner sheet.

$\theta-\theta'$ varies so slowly that it seemed sufficient for the purposes of comparison to calculate every fifth theoretical value, with the exception of that portion of Table V. for which the variations of $\theta-\theta'$ are sensible.

The section cuts the principal section A O C between lines 4 and 5, Table V. Just at that point the theoretical value is greater than the experimental. It continues so until we reach line 18, Table V., that is for an arc of about 9°.

From that point onwards the theoretical value of μ_2 is uniformly less than the experimental.

Thus the result of experiment for the outer sheet is represented by a curve which coincides with the theoretical section at the point in which it cuts the plane A O C, then lies inside it for an arc of about 9°.

At this point the two again coincide, and for the rest of the arc experimented on, the section given by experiment lies outside that given by theory; the difference between the two increasing throughout Tables VI. and VII., and diminishing again in Table VIII.

Section VII.—*Effect of possible Errors in the Values of μ_a, μ_b, μ_c discussed.*

The next task will be to determine the effect of any small errors which may have occurred:

(1.) In the determination of μ_a, μ_b, μ_c;

(2.) In the angles which fix the position of the planes P R, P Q relatively to the axes of the surface.

(1.) μ_a. To determine μ_a we considered the length of the radius vector common to the principal section A O C and the section by the plane P R. This radius vector is fixed in length, and is on FRESNEL'S theory the radius vector of an ellipse, axes μ_a and μ_c, inclined at a small angle to μ_a.

The increase in the value of μ_a will therefore increase the value of this angle, that is, it will alter the position of the plane P L R′ and be more properly considered under (2).

μ_b is determined by the intersection of the section of the inner sheet by the plane P R, with a circle of radius μ_b, and is therefore fixed at least within the limits of experimental error.

μ_c is found from an independent and, on the whole, less trustworthy series of observations—less trustworthy because the faces were much less plane than P, Q, R.

Let us consider the effect of decreasing the value of μ_c.

If ϵ be the angle between the optic axes

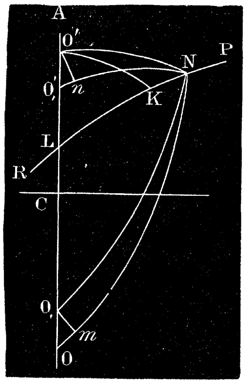
$$\tan\epsilon = \sqrt{\left\{\frac{\dfrac{1}{\mu_b^2}-\dfrac{1}{\mu_a^2}}{\dfrac{1}{\mu_c^2}-\dfrac{1}{\mu_b^2}}\right\}}$$

As μ_c decreases $\tan\epsilon$ decreases; therefore ϵ decreases.

Let us consider how this affects θ and θ'.

Taking the same figure as previously.

Fig. 12.



Let O' K be an arc perpendicular to A C.
Let $O_{\prime}$ $O_{\prime}'$ be the new positions of O O'.
Then

$$O_{\prime}O=O_{\prime}'O'=\delta\epsilon$$

Let $O_{\prime}$ m, O' n be perpendicular on N O, N $O_{\prime}'$ respectively.

$$O'L=7^{\circ} \text{ approximately}$$
$$O'LP=60^{\circ} \text{ approximately}$$
$$\therefore LK=14^{\circ} \text{ approximately}$$

Let N fall between P and K

$$\delta\theta=-Om=-\delta\epsilon\cos NOL$$
$$\delta\theta'=O_{\prime}'n=\delta\epsilon\cos NO_{\prime}'A$$
$$\delta(\theta+\theta')=-\delta\epsilon\,(\cos NOL-\cos NO_{\prime}'A)$$

Thus $\delta(\theta+\theta')$ is negative and decreases numerically as we approach P; for N O L, N $O_{\prime}'$ A become more nearly equal.

Thus between K and P $\theta+\theta'$ decreases with ϵ.

Between L and K both θ and θ' decrease, therefore $\theta+\theta'$ decreases *à fortiori;* but $\delta(\theta+\theta')$ is always numerically less than $2\delta\epsilon$.

Let us see how these changes affect μ_1.

We have [Section IV. (1)]

$$\frac{2}{\mu_1^2}=\frac{1}{\mu_a^2}+\frac{1}{\mu_c^2}-\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta+\theta').$$

If μ_c is decreased, $\frac{1}{\mu_c^2}+\frac{1}{\mu_a^2}$ is increased, $\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}$ is increased, $\theta+\theta'$ is decreased.

Therefore $\cos(\theta+\theta')$ is increased so long as $\theta+\theta'$ is less than 90°.

Therefore $\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta+\theta')$ is increased so long as $\theta+\theta'$ is less than 90°.

And the increase in $\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta+\theta')$ is greatest when $\theta+\theta'$ is least.

Also the increase in $\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta+\theta')$ is less than the increase in $\left(\frac{1}{\mu_c^2}+\frac{1}{\mu_a^2}\right)$.

Therefore $\frac{1}{\mu_c^2}+\frac{1}{\mu_a^2}-\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta+\theta')$ is increased, and this increase is greatest when $\theta+\theta'$ is greatest.

Therefore, as $\theta+\theta'$ increases up to 90° μ_1 decreases.

When $\theta+\theta'$ is greater than 90°, $\cos(\theta+\theta')$ is negative.

As $\theta+\theta'$ increases, $\cos(\theta+\theta')$ increases numerically.

Therefore $\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta+\theta')$ increases numerically and is negative.

Therefore $\frac{1}{\mu_c^2}+\frac{1}{\mu_a^2}-\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta+\theta')$ increases *à fortiori.*

Therefore μ_1 decreases.

Thus the effect of decreasing μ_c is to produce a decrease in the value of μ_1, and this decrease continually increases as we go from L to Q.

Now if we compare the differences between the theoretical and experimental values of μ_1, given in Tables V. and VI., we see that they increase with $\theta+\theta'$, the theoretical value being in excess of the experimental.

The effect of the assumed alteration in the value of μ_c would be to decrease the theoretical value of μ_1 by an amount which constantly increases, that is, to bring it more nearly into agreement with experiment.

But the effect in Tables VII. and VIII. would be contrary.

On referring to them we see that the differences between theory and experiment continually decrease.

The effect of the proposed alteration would be to subtract from the theoretical value of μ_1 given in those tables: a quantity which increases rapidly as we get further from L.

The result would be that the theoretical value would soon become less than the experimental, and the difference between the two would continually increase.

We might, it is true, choose the decrement of μ_c so as to make the extremities of the arcs considered coincide, and then the differences would be diminished throughout. The effect of this would be to make the theoretical section lie outside of the experimental for about 75°, then cut it and lie inside it.

Whether the two could again be made to coincide when cutting the plane A O B experiment alone could settle.

Let us now consider the effect of the supposed change on μ_2.

We have

$$\frac{2}{\mu_2^2}=\frac{1}{\mu_a^2}+\frac{1}{\mu_c^2}-\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta-\theta').$$

We have seen that θ is decreased by the change.

θ' is at first decreased, though by less than θ, afterwards it is increased.

Therefore, for any point on the curve L P the value of $\theta-\theta'$ is decreased by this alteration.

Therefore $\cos(\theta-\theta')$ is increased.

$\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta-\theta')$ is increased, though by less than $\left(\frac{1}{\mu_c^2}+\frac{1}{\mu_a^2}\right)$.

Therefore $\frac{1}{\mu_c^2}+\frac{1}{\mu_a^2}-\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta-\theta')$ is increased.

Therefore μ_2 is decreased.

Or the variation in μ_2 is in the same direction as in μ_1.

But since $\theta-\theta'$ is $<\theta+\theta'$ $\quad\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta-\theta')$ is $>\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta+\theta')$.

And the whole change in

$$\frac{1}{\mu_c^2}+\frac{1}{\mu_a^2}-\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta-\theta')$$

is much less than the whole change in

$$\left(\frac{1}{\mu_c^2}+\frac{1}{\mu_a^2}\right)-\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta+\theta').$$

Or the decrease in μ_2 is small compared with that in μ_1.

Again, as the variation in $\theta-\theta'$ varies from zero at L to a maximum about K, while $\cos(\theta-\theta')$ also varies considerably, the variation in μ_2 will be considerably different for different points between L and K.

After passing K, however, $\theta-\theta'$ is nearly constant all along the curve.

The change in the value of μ_2, therefore, will not differ much for different points, that is, for different waves.

The whole effect of the alteration, therefore, will be to decrease throughout the theoretical value of μ_2, this decrease being greatest about Table V., line 23.

In the greater part of Table V. the result will be to produce a closer agreement between theory and experiment.

But in the rest of the work, since the theoretical value of μ_2 is less than the experimental, the effect will be to increase the difference between the two and so widen the discrepancy between theory and experiment.

Thus taking both sheets we cannot produce on the whole a closer agreement between theory and experiment by decreasing the value of μ_c. Neither can we by increasing it.

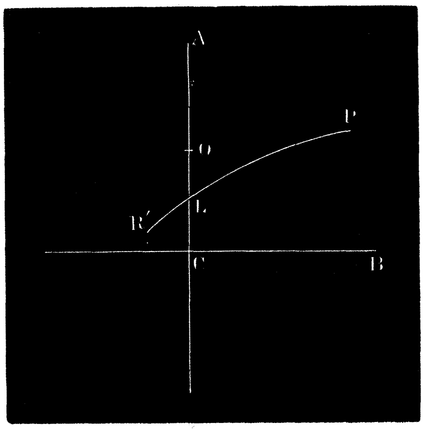
For though we might produce closer agreement for the outer sheet by thus increasing the theoretical value of μ_2, we should add greatly to the error in the case of the inner sheet.

Section VIII.—*Effect of Change in Position of Planes of Prisms with reference to the Principal Planes of the Crystal.*

We have now to discuss the effect of small variations in the position of the planes P R and P Q.

We take P R first, and we consider its position as determined by C L and the angle A L P, while the position of the normals to P and R is fixed by P L (*vide* Section II.).

Fig. 13.



Now variations in C L and A L P produce variations in O L, O′ L, λ λ′, &c., and hence in θ θ', while variations in L P only change θ and θ', O L, O′ L, λ and λ′ remaining constant.

Let us consider, therefore, the variations of L P first.

A reference to our tables shows us that through the greater part of the arcs L P, P Q′, $\theta-\theta'$ is constant, so that $\delta\theta$ considered as dependent on L P is equal to $\delta\theta'$ nearly.

Again,

$$\frac{2}{\mu_1^2}=\left(\frac{1}{\mu_a^2}+\frac{1}{\mu_c^2}\right)-\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta+\theta')$$

$$\frac{2}{\mu_2^2}=\left(\frac{1}{\mu_a^2}+\frac{1}{\mu_c^2}\right)-\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta-\theta')$$

and since the change in $(\theta-\theta')$ due to a small increment of L P is very small, such an increment will not alter the value of μ_2.

But if $\theta-\theta'$ is nearly constant for a small motion along L P, $\theta+\theta'$ will be nearly so for a small motion perpendicular to L P.

Thus a small motion perpendicular to P L will not alter appreciably the value of μ_1;

we may therefore obtain corrections to μ_1 μ_2 independently, to μ_1 by supposing an increase in the value of L P, to μ_2 by supposing the planes P R, P Q to turn through a small angle about some line in them.

Moreover, we shall consider that line to be in each case the line of intersection with the plane A O C, and treat this line as the same for both planes, the two lines thus considered as coincident are really inclined at an angle of 3′.

To find then the variation which we must assume in L P to bring the results of theory and experiment into closer agreement for the inner sheet, we proceed as follows—

If in the equation

$$\frac{2}{\mu_1^2}=\left(\frac{1}{\mu_a^2}+\frac{1}{\mu_c^2}\right)-\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta+\theta')$$

we substitute an experimental value for μ_1, we can determine a value of $(\theta+\theta')$. Subtracting this from the calculated value, we get $\delta(\theta+\theta')$.

But for the displacement considered $\delta\theta=\delta\theta'$.

Hence we have found $\delta\theta$; but $\cos(\text{LN}-\lambda)=\frac{\cos\lambda}{\cos \text{OL}}\cos\theta$. O L and λ are unaltered by the displacement supposed; therefore we get δ(L N), and L P=L N+ϕ'. ϕ' is constant, therefore we have found δ L P.

To put this plan into execution, I chose as the experimental value for μ_1 the 21st in Table VI.

$$\mu_1=1{\cdot}62807$$

the differences between theory and experiment having at that point a maximum value and being fairly regular.

The resulting value for $(\theta+\theta')$ is

$$70^\circ\ 57'\ 20''$$

The calculated value is

$$70^\circ\ 16'\ 40''$$

The difference is

$$0^\circ\ 40'\ 40''$$

$$\therefore\ \delta\theta=0^\circ\ 20'\ 20''$$

This gives for δ(L P)

$$\delta(\text{L P})=0^\circ\ 21'\ 20''$$

Thus at this point the result of theory and experiment could be brought into coincidence by supposing the line O P normal to the face P to turn through an angle of 0° 21′ 20″ in the plane L P away from L.

Before discussing the effect of this change on the values of μ_1 in other directions, let us consider the variation it will be necessary to make in the angle A L N to bring the theoretical results for the outer sheet into accordance with experiment. Taking the experimental value for μ_2 corresponding to the incident wave which gave rise to the value of μ_1 we have just been discussing, we have

$$\mu_2=1\cdot68447$$

Whence by means of the formula

$$\frac{2}{\mu_2^2}=\left(\frac{1}{\mu_a^2}+\frac{1}{\mu_c^2}\right)-\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta-\theta')$$

we find

$$\theta-\theta'=9^\circ\ 15'$$

The value of $\theta-\theta'$ given by theory is Table VI., line 21,

$$\theta-\theta'=9^\circ\ 30'$$

We have to find the change in the value of χ or A L N which will produce this. In this variation $\theta+\theta'$ is constant

$$\therefore\ \delta\theta=-\delta\theta'$$

but

$$\delta(\theta-\theta')=-15'$$

$$\therefore\ \delta\theta=-7'\ 30''$$

$$\delta\theta'=\ \ 7'\ 30''$$

From these results we can find the increment in χ from the equations

$$\cos\theta=\frac{\cos \text{OL}}{\cos\lambda}\cos(\text{LN}-\lambda)$$

$$\tan\lambda=\tan \text{OL}\cos\chi$$

$$\cos\theta=\cos \text{OL}\cos \text{LN}+\sin \text{OL}\sin \text{LN}\cos\chi$$

$$\therefore\ \delta\chi=\frac{\sin\theta\delta\theta}{\sin \text{OL}\sin \text{LN}\sin\chi}$$

Substituting values we are led to a value of $\delta\chi$, nearly equal to 1°. So that the new value of $\chi=60^\circ\ 20'$ [Section II. (17)].

This change in the value of χ produces a corresponding change in the values of λ, λ'. The new values are for the prism P R

$$\lambda=5^\circ\ 12'\ 20'' \quad \ldots\ldots \quad (1)$$

$$\lambda'=3^\circ\ 50'\ \ 0'' \quad \ldots\ldots \quad (2)$$

while

$$\log \frac{\cos \text{OL}}{\cos \lambda} = \bar{1}\cdot 9945594 \quad . \quad . \quad . \quad . \quad (3)$$

$$\log \frac{\cos \text{OL}}{\cos \lambda'} = \bar{1}\cdot 9970325 \quad . \quad . \quad . \quad . \quad (4)$$

for the prism P Q

$$\lambda_{,} = 5^{\circ}\ 10'\ 20'' \quad . \quad . \quad . \quad . \quad (5)$$

$$\lambda_{,}' = 3^{\circ}\ 54' \quad . \quad . \quad . \quad . \quad (6)$$

$$\log \frac{\cos \text{OL}'}{\cos \lambda_{,}} = \bar{1}\cdot 9946764 \quad . \quad . \quad . \quad . \quad (7)$$

$$\log \frac{\cos \text{O}'\text{L}'}{\cos \lambda_{,}'} = \bar{1}\cdot 9969629 \quad . \quad . \quad . \quad . \quad (8)$$

The ensuing Table gives the values of μ_1, μ_2 for nine different incident waves taken from the previous work, calculated on the supposition that both these changes have been made in the position of the plane P R.

The column headed Difference gives the excess of theory over experiment; that headed α gives the excess of the theory before it was modified; β gives the change in the theoretical value of μ_1 due to this displacement.

Table IX.

Outer sheet.			Inner sheet.		
Where taken from Experiment Table.	$\theta-\theta'$.	LN + λ.	LN − λ'.	$\theta+\theta'$.	
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
Table V., line 8	4 25 20	7 3 30	1 57 30	18 25 40	1
,, ,, ,, 18	8 20 50	14 12 40	5 10 20	25 14 50	2
,, ,, ,, 28	9 12 0	24 6 10	15 3 50	42 6 0	3
,, VI. ,, 11	9 15 20	22 4 20	23 2 0	57 7 40	4
,, ,, ,, 21	9 14 30	39 14 0	30 11 40	70 57 30	5
,, VII. ,, 11	9 12 30	46 31 0	37 26 50	85 8 30	6
,, ,, ,, 21	9 9 50	56 16 30	47 12 20	90° + 14 18 50	7
,, VIII. ,, 11	9 5 40	71 3 0	61 59 0	,, 43 29 0	8
,, ,, ,, 26	9 4 20	80 39 20	71 35 10	,, 62 30 40	9

Difference.	μ_2 New Theory.	μ_2 Experiment Value.	μ_2 Former Theory.	μ_1 Former Theory.	μ_1 Experiment Value.	μ_1 New Theory.	Difference.		α.	β.
+ 7	1·68533	1·68526	1·68537	1·68106	1·68099	1·68012	+ 3	1	+ 7	4
+11	1·68465	1·68454	1·68462	1·67743	1·67721	1·67707	−14	2	+22	37
− 3	1·68445	1·68448	1·68437	1·66352	1·66300	1·66285	−15	3	+52	67
− 9	1·68443	1·68452	1·68437	1·64674	1·64603	1·64590	−13	4	+71	86
− 4	1·68443	1·68447	1·68437	1·62897	1·62807	1·62806	− 1	5	+90	91
− 9	1·68444	1·68453	1·68438	1·60974	1·60897	1·60882	−15	6	+77	92
−12	1·68445	1·68457	1·68439	1·58432	1·58365	1·58346	−19	7	+67	86
− 5	1·68447	1·68452	1·68440	1·55216	1·55157	1·55138	−19	8	+59	78
− 4	1·68448	1·68444	1·68442	1·53793	1·53774	1·53779	+ 5	9	+29	24

It will be noticed that the differences for this modified theory are much less than for the original, but they have nearly all changed sign. In fact, the correction applied, calculated so as to produce the closest possible agreement in line 5, has been too great, and the differences can be reduced by applying a similar but smaller correction in the opposite direction.

Now this variation in the value of μ_1 has been caused almost entirely by the increase of the angle L P; we must therefore decrease this angle again, and as the numbers in the difference column are each about one-fifth of the corresponding number in the column β, we must decrease L P by one-fifth of the amount by which it was previously increased.

But this increase was

$$\delta(LP)=0° \, 21' \, 20''$$

we must therefore decrease this by

$$0° \, 4' \, 15'$$

leaving the total increase

$$0° \, 17' \, 5'$$

In consequence of this, we must add to the modified theoretical values of μ_1 respectively, quantities equal to one-fifth of the corresponding numbers in column β.

The result gives us a new set of theoretical values, which agree with experiment throughout to a remarkable degree.

These results are expressed in Table X.

Column 1 gives the values of μ_1 already modified.

Column 2 gives one-fifth of quantities in β, which, when added to the values of μ_1 in Column 1 respectively, give Column 3 the resulting theoretical values.

Column 4 gives the experimental values, and Column 5 the excess of theory over experiment.

TABLE X.

Modified value of μ_1.	$\frac{1}{5}\beta$.	Final value of μ_1.	Experimental value of μ_1.	Difference.	
1·68102	1	1·68103	1·68099	4	1
1·67707	7	1·67714	1·67721	−7	2
1·66285	13	1·66298	1·66300	−2	3
1·64590	17	1·64607	1·64603	4	4
1·62806	18	1·62824	1·62807	17	5
1·60882	18	1·60900	1·60897	3	6
1·58346	17	1·58363	1·58365	−2	7
1·55138	16	1·55154	1·58157	−3	8
1·53779	5	1·53784	1·53774	10	9

These differences are exceedingly small—sometimes positive, sometimes negative—but rarely greater than the possible error of experiment as appears from the tables of error in I., II., III., and IV.

Thus we have found a plane section of the surface which agrees closely with the results of experiment.

This section passes through L, a point on the principal section A C at an angular distance of 1° 21′ 42″ from C, is inclined to that section at an angle of 60° 20′, while the normal to the face P lies at an angular distance from L equal to

$$35^\circ\ 0'\ 19''+0^\circ\ 17'\ 15''=35^\circ\ 17'\ 34''$$

Under these circumstances it will be necessary to consider the probability of making such an error as this result indicates in the position of the normal to P with reference to the principal axes.

But before doing this we must say a few words about the variations in the values of μ_2 produced by the correction now considered.

The final correction will not affect any of them except the first, which will be slightly increased by it.

This follows from the fact that between lines 1 and 2 of Table IX. $\theta-\theta'$ varies considerably, and increases or decreases with $\theta+\theta'$, so that μ_1, μ_2 increase or decrease together.

The change considered has increased μ_1, it will therefore increase μ_2.

The difference in line 2 is considerable, but it may be noticed that the experimental value is clearly too small, falling as it does (Table I., line 18) between two considerably greater values. In the other cases the differences between theory and experiment have been diminished but remain of the same sign as before.

A still further displacement in the same direction would therefore produce still further agreement.

We have now to consider the effect of varying C L.

Let us trace the effect of increasing C L by 10′.

The value of μ_a depends on that of C L being found from it by the formula (Section V.)

$$\frac{1}{\mu_a^2}=\sec^2 \mathrm{CL}\left(\frac{1}{r^2}-\frac{\sin^2\mathrm{CL}}{\mu_c^2}\right)$$

where

$$\mu_c^2=1\cdot53013$$

and r is given from experiment

$$r=1\cdot68550$$

The new value of a will be found to be

$$\mu_a=1\cdot68571$$

instead of 1·68560

This increase in the value of a will increase the angle betwen the optic axes from 9° 4′ 5″, the value already found, to 9° 9′ 20″.

We have, therefore, to discuss the effect of these changes in the theoretical values of μ_1, μ_2.

We will further suppose that the angle χ is so altered that the section still passes through the former position of P—*i.e.*, so that the only error made in the previous determination of its position has been in the position of R′.

This is probable, for C R′ is small and A R′, B R′, nearly right angles.

The new value of χ will be

$$\chi = 59^\circ\ 32'$$

These changes will produce variations in the values of O L, O′L, $\lambda\ \lambda'$, for P R′, and O L′, O′ L′, $\lambda_{,}$ $\lambda_{,}'$, for P Q (we assume that L L′ is unaltered).

The complexity of the changes rendered it very difficult to discuss the effects on μ_1, μ_2 from general considerations. I have therefore calculated the new values in five different positions taken from the previous tables.

The table following gives the general results.

TABLE XI.

Whence taken.	Difference.	μ_1 Modified Theory.	μ_1 Experiment.	μ_1 Former Theory.	Difference.	
Table V., line 28	36	1·66386	1·66300	1·66352	52	1
,, VI., ,, 11	49	1·64652	1·64603	1·64674	71	2
,, VI., ,, 21	65	1·62872	1·62807	1·62897	90	3
,, VII., ,, 21	41	1·58406	1·58365	1·58432	67	4
,, VIII., ,, 11	43	1·55200	1·55157	1·55216	59	5

Whence taken.	Difference.	μ_2 Modified Theory.	μ_2 Experiment.	μ_2 Former Theory.	Difference.	
Table V., line 28	−1	1·68447	1·68448	1·68437	−11	1
,, VI., ,, 11	−6	1·68446	1·68452	1·68437	−15	2
,, VI., ,, 21	+1	1·68448	1·68447	1·68437	−10	3
,, VII., ,, 21	−7	1·68450	1·68457	1·68439	−18	4
,, VIII., ,, 11	0	1·68452	1·68452	1·68440	−12	5

The difference table on the left is between the modified theory and experiment, that on the right between the former theory and experiment.

The upper table refers to the inner, the lower to the outer sheet.

It will be seen that the differences for μ_1, μ_2 are both decreased, though still remaining of the same sign.

The differences for μ_1 are about two-thirds of their former value.

If we then increase C L still further by twice as much as previously, so that the total increase is 30′, we shall get still closer agreement for μ_1.

TABLE XII.

α.	β.	μ_1 Final Theory.	μ_2 Experiment.	Difference.
16	48	1·66304	1·66300	4
22	66	1·64608	1·64603	5
25	75	1·62822	1·62807	15
26	78	1·58354	1·58365	−11
16	48	1·55268	1·55157	11

γ.	δ.	μ_2 Final Theory.	μ_2 Experiment.	Difference.
10	30	1·68467	1·68448	19
9	27	1·68464	1·68452	12
11	33	1·68470	1·68447	23
11	33	1·68472	1·68457	15
12	36	1·68475	1·68452	23

Column α gives the change in μ_1 due to a change of 10′ in C L.

Column β gives three times that change which, assuming the theory of proportional parts, will be the change in μ_1 due to a change of 30′ in C L.

Subtracting this from the values of μ_1 given by the original theory, we get the final theoretical values of μ_1. These agree much more closely with experiment than the original values.

But the effect of this further change will be to over correct μ_2.

The first change has reduced the differences for μ_2 by the amounts given by γ, Table XII.

Multiplying these by 3 we get δ, the changes produced in μ_2 by a change of 30′ in C L, and adding these values to the original theoretical values of μ_2 we get the final theory, which differs from experiment by rather more than the original theory and in the other direction.

Thus an increase of C L will not, on the whole, produce the required effect. Similarly a decrease will not produce it either.

Section IX.—*Possibility of an Error in the Positions of the Faces discussed.*

We have seen, however, that by increasing L P by 17′ 5″ and A L P by 1°, we can make the agreement between theory and experiment extremely close.

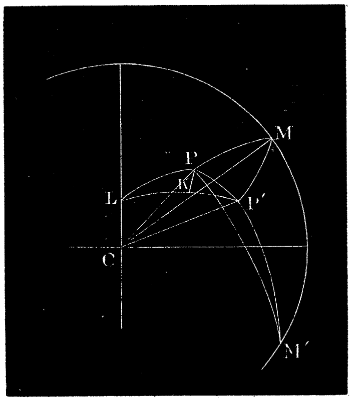
This alteration in the values of L P and A L P may be effected in two ways.

We may suppose either that, the axes of elasticity retaining their position with reference to the planes m, m', we have made an error in the observed angles P M, P M′ (Section II.), and hence in the position of P referred to the axes.

Or, secondly, that the position of P relatively to m m' has been accurately found, but that the axes of elasticity differ slightly in direction from their assumed positions, so that O C is not parallel to the intersection of m m', and O B does not bisect the angle M O M′.

We will take the first hypothesis first, and consider what must be the error in P M, P M′ to give rise to the required change in the position of P.

Fig. 14.



Let P′ be the new position of P, we require M P′ M′ P′.

Let P K be perpendicular on L P′

$$\begin{aligned} LK &= LP \text{ approximately} \\ KP' &= 0^\circ\ 17' \text{ approximately} \\ KP &= PLK \sin 35^\circ \text{ approximately} \\ &= 60' \times {\cdot}57 \\ &= 34' \text{ approximately} \end{aligned}$$

Now P K P′ is approximately a small plane triangle with a right angle at K.

$$\begin{aligned} PP' &= \sqrt{PK^2 + P'K^2} \\ &= \sqrt{34^2 + 17^2} \\ &= 17'\sqrt{5} \end{aligned}$$

whence

$$PP' = 38'$$

Again

$$\cos CP' = \cos CL \cos LP' - \sin CL \sin LP' \cos ALP'$$

whence

$$\begin{aligned} CP' &= 35^\circ\ 58' \\ CP &= 35^\circ\ 43' \quad [\text{Section II. } (12)] \\ PP' &= 0^\circ\ 38' \end{aligned}$$

Hence from the triangle P C P′ we get

$$PCP' = 0° \ 58$$

But from the triangle A C P

$$ACP = 57° \ 42' \ 30'' \quad \text{[Section II.]}$$

$$\therefore ACP' = 58° \ 40' \ 30''$$

but

$$ACM = 58° \ 7' \ 30'' \quad \text{[Section II. (7)]}$$

$$\therefore MCP = 0° \ 33'$$

$$\cos MP' = \cos MCP \sin CP'$$

whence

$$MP' = 54° \ 58' \ 50''$$

The original value of M P′ was as given by the mean of a large number of observations

$$MP' = 54° \ 17' \ 6'' \quad \text{[Section II. (3)].}$$

The difference, amounting as it does to over 40′, is far beyond any possible error of experiment.

Let us find further the change in M′ P,

$$\cos M'P' = \sin CP' \cos M'CP'$$

$$M'CP' = 63° \ 47' - 33'$$

$$= 63° \ 14'$$

whence

$$M'P' = 74° \cdot 39' \ 40''$$

The observed value was

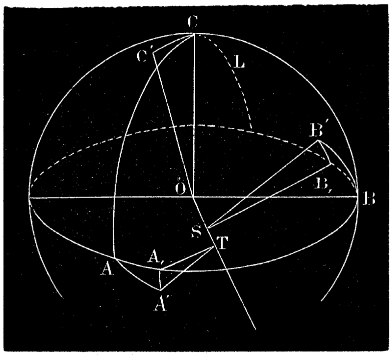
$$M'P = 75° \ 16' \ 20'' \quad \text{[Section III. (4)].}$$

The difference is 0° 36′ 40″. This, again, is far beyond any possible experimental error. Thus, so long as we assume the position of the axes of elasticity to be definitely fixed in the crystal so that O C is parallel to the intersection of *m m′* while O B bisects the angle between them, the displacement of the plane P R′ necessary to bring the results of experiment and theory into agreement is far too great to be possible.

But there remains the supposition that the axes of elasticity have not exactly the same position in all crystals of arragonite. So that the displacement of P R relatively to the axes might be effected by changing slightly their position with reference to the faces of the crystal, P R retaining their position, relatively to those faces, unchanged. The possibility of this is a question for the mineralogist. I have been as yet unable to find data for a satisfactory answer. It seems, however, plausible to suppose that in a substance like arragonite which is not chemically pure, but contains foreign substances to a variable degree in different specimens, some slight variation such as that indicated might occur.

We can, without much difficulty, calculate the amount of the change.

Fig. 15.



We may very approximately treat the rotation round O L as if it were about O C, C L being very small.

The effect of a rotation of 1° about O C will be to bring A and B to $A_{,}$ $B_{,}$ respectively in the same plane, where

$$AA_{,}=BB_{,}=1^{\circ}$$

$A_{,}$ $B_{,}$ being to the right of A, and B.

The increment of 17′ in L P may be effected by turning the axes through an angle of 17′ about a normal to the plane L P passing through O, let O S T be this normal, let A′ B′ C′ be the final positions of A B C.

$A_{,}A'$, $B_{,}B'$ are arcs of small circles whose centres lie on O S T at T and S respectively, so that

$$A_{,}TA'=B_{,}SB'=17'$$

C C′ is an arc of a great circle, and is $=17'$.

$$B_{,}B'=SB_{,}\times B'SB_{,}$$

$$SOB_{,}=60^{\circ} \text{ approximately}$$

$$\therefore SB'=\sin 60^{\circ}=\frac{\sqrt{3}}{2}$$

$$\therefore B_{,}B'=\frac{17'\sqrt{3}}{2}$$

$$BB'=\sqrt{(BB_{,}^{2}+B'B_{,}^{2})}$$

$$=\sqrt{\left(60'^{2}+\frac{3\times 17^{2}}{4}\right)}$$

$$=1^{\circ}\ 2' \text{ approximately}$$

$$A_{,}A'=ATA' \sin 30$$

$$=8'\ 30''$$

$$AA'^{2}=\sqrt{\left(60'^{2}+\frac{17'^{2}}{4}\right)} \text{ approximately}$$

$$=1^{\circ}\ 1' \text{ approximately.}$$

Thus the new positions of the axes are inclined to the old at angles of 1° 1′, 1° 2′, and 17′ respectively for O A, O B, O C. So that the axis of C in its new position would be inclined to the line of junction of $m\ m'$ at an angle of 17′, while that of b would be inclined to the bisector of these planes at an angle of 1° 2′.

If variations such as here considered be possible, so that we cannot be certain *à priori* of the position of the axes of elasticity, the only method of testing FRESNEL's or any other theory will be by trial.

Section X.—*General Results of the Investigation.*

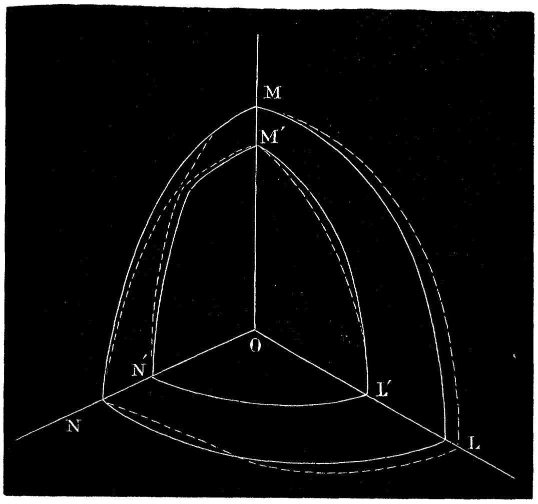
Combining the results of the two series of experiments, it seems to me most probable that FRESNEL's theory is only true as a first approximation.

Both series of observations have led to the discovery of considerable deviation from the theory, unless we assume errors in the experiments, especially in the determination of the position of the planes of the prisms with reference to the axes of the crystals, which are greatly in excess of the amounts we can reasonably expect.

It will be noticed that the values taken for the constants μ_a and μ_b in the two parts of the paper differ by ·0002 and ·00017 respectively. This is due to the fact that the two crystals used were different. RUDBERG (POGG. Annalen, vol. xvii., p. 1) found differences of as much as ·0004 in the values of μ_b deduced from two specimens of arragonite.

It may be objected that the variations between theory and experiment are not in exactly the same direction in the two Parts; but we must remember (1) that the arcs investigated are taken from entirely different portions of the surface; (2) in the first Part, the approximately elliptic section belongs to the outer sheet; in the second, to the inner sheet of the wave surface.

Fig. 16.



The accompanying figure will illustrate the results of the investigation.

M O N is the principal plane of the first prism nearly coincident with the plane through the optic axes A O C.

N O L the principal plane of the second prism nearly coincident with the plane B O A, so that O N is nearly coincident with O A.

M O L the principal planes of the two prisms in the second crystal here treated as coincident, inclined at about 60° to B O C, and cutting it in a line nearly coincident with O C.

Hence O M is not far removed from O C. O L is nearly in the plane A O B and inclined at about 60° to O A or O N.

The strong lines give approximately the form of the sections of FRESNEL'S surface by these planes, the dotted lines the results of experiment.

In the case of the arc N′ L′ the results of theory and experiment agreed closely.

For the arcs M N, M′ N′ the experiments covered an arc of about 16° from M and M′.

For N L the experiments covered an arc extending from L to about 10° on the side nearest to N of the point where the two arcs intersect.

For M L the experiments extended over an arc of about 70° measured from M.

Section XI.—*Effect of Dispersion considered.*

The theory of dispersion appears to me to afford a more probable explanation of these small variations from FRESNEL'S construction.

FRESNEL himself remarked ('Second Supplément au premier Mémoire sur la double réfraction,' Œuvres Complètes de FRESNEL, tom. ii.) that in the case of the vibrations which constitute light the radius of the sphere of action of the molecular forces brought into play by the vibration is not necessarily very small compared with the wave length.

And, consequently, it is incorrect to suppose that the propagation of each of the disturbances of which a vibration is composed is uninfluenced by the disturbances which precede and follow it, and that the velocity of propagation is independent of the manner in which they proceed and follow it.

This supposition is the basis of FRESNEL'S work on double refraction.

Let us consider the effect of dispersion in a doubly refracting medium.

In an isotropic medium the relation between V, the velocity of wave propagation, and λ, the wave length, is generally allowed to be

$$\frac{1}{V}=a+\frac{b}{\lambda^2}+\frac{c}{\lambda^4}+, \&c.,$$

a, b, c, &c., being constant, the values of the terms continually decreasing, so that except in highly dispersive media we may put $\frac{1}{V}=a+\frac{b}{\lambda^2}$ with sufficient exactness.

Let us suppose that an equation of this form holds in crystalline media also, only that a, b, c, &c., instead of being constants are functions of the directions of propagation and vibration; and, further, let us suppose that FRESNEL'S construction is true for waves of infinite length, so that the equation $\frac{1}{v}=a$ gives us a FRESNEL'S wave surface.

From the known values of the constants of the wave surface for different rays of the spectrum the constants of the surface for infinite wave lengths can be found, and hence the value of a calculated in any given direction.

If from experiment we find the value of $\frac{1}{v}$ or μ for any wave length, the difference $\mu-a$ ought on this theory to be equal to $\frac{b}{\lambda^2}$.

And if we find the values of μ for different wave lengths (μ_1, μ_2, μ_3 say) in the same direction, we have

$$\mu_1-a=\frac{b}{\lambda_1^2}$$

$$\mu_2-a=\frac{b}{\lambda_2^2}$$

$$\mu_3-a=\frac{b}{\lambda_3^2}$$

whence

$$\frac{\mu_1-a}{\mu_2-a}=\frac{\lambda_2^2}{\lambda_1^2}$$

$$\frac{\mu_2-a}{\mu_3-a}=\frac{\lambda_3^2}{\lambda_2^2}$$

To verify this I observed the values of μ in two directions for the rays C, D, and F, with the following results :—

$$\left(\frac{\lambda_C}{\lambda_D}\right)^2=1\cdot2403$$

$$\frac{\mu_D-a}{\mu_C-a}=1\cdot2875 \text{ (first experiment)}$$

$$1\cdot2770 \text{ (second experiment)}$$

$$\left(\frac{\lambda_D}{\lambda_F}\right)^2=1\cdot46978$$

$$\frac{\mu_F-a}{\mu_D-a}=1\cdot47208 \text{ (first experiment)}$$

$$=1\cdot47348 \text{ (second experiment).}$$

The numbers, especially in the last case, are sufficiently close to make it seem worth while continuing the investigations.